Iteration of mapping classes on a Bers slice: examples of algebraic and geometric limits of hyperbolic 3-manifolds

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Abstract. We present two examples of sequences of quasi-Fuchsian groups whose algebraic limit Γ_A is a proper subgroup of its geometric limit Γ_G. Each arises from iteration of an element of the mapping class group $\mathrm{Mod}(S)$ of a surface S. In the first example (developed by Kerckhoff and Thurston) iteration of a Dehn twist $\tau \in \mathrm{Mod}(S)$ determines a sequence of groups that develop a new rank-2 parabolic subgroup in their geometric limit. In the second example, which arises from iteration of a mapping class $\varphi \in \mathrm{Mod}(S)$ with both pseudo-Anosov and finite order restrictions, the groups develop a new totally degenerate subgroup in their geometric limit. Thurston's pleated surfaces play an essential role in our analysis.

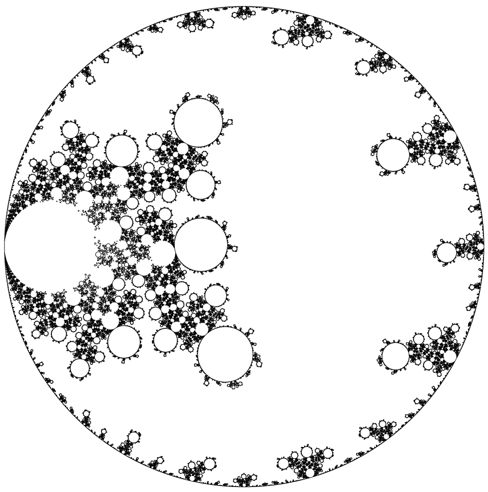

Figure 1. A partially degenerate group on Bers' boundary.

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1. Introduction

In the late eighties, work of Jørgensen, Marden, Kerckhoff and Thurston revealed that the study of algebraic and geometric limits of Kleinian groups and how they may differ is an important tool for studying deformation spaces of hyperbolic 3-manifolds. In particular, Interesting phenomena arise when the algebraic limit sits as a *proper* subgroup of the geometric limit. In this paper we will discuss various ways in which this can occur in the setting of Kleinian groups isomorphic to the fundamental group of a surface.

A sequence

$$\rho_n : G \to \mathrm{PSL}_2(\mathbb{C})$$

of faithful representations of a group G with discrete image is said to converge *algebraically* if they converge in the compact open topology - the images of finite generating sets for G under ρ_n converge in $\mathrm{PSL}_2(\mathbb{C})$.

Another notion of convergence of Kleinian groups comes from the *Hausdorff topology:* we may also examine convergence of the images $\{\rho_n(G)\} = \Gamma_n$ in the Hausdorff topology on closed subsets of $\mathrm{PSL}_2(\mathbb{C})$, or the *geometric topology.* A sequence of Kleinian groups $\{\Gamma_n\}$ converges in this sense to a Kleinian group Γ_G called its *geometric limit* if each of the following two conditions is satisfied.

1. For each $\gamma \in \Gamma_G$, there are $\gamma_n \in \Gamma_n$, such that $\gamma_n \to \gamma$ in $\mathrm{PSL}_2(\mathbb{C})$, and
2. If there exists $\{\gamma_{n_i}\}$ such that $\gamma_{n_i} \to \gamma$ in $\mathrm{PSL}_2(\mathbb{C})$ then $\gamma \in \Gamma_G$.

Note that condition (2) is automatically satisfied for algebraic limits on a finite generating set, so given an algebraic limit ρ_∞ of an algebraically convergent sequence of representations $\{\rho_n\}$, the image $\rho_\infty(G)$ is a subgroup of any geometric limit Γ_G. When $\rho_\infty(G) = \Gamma_G$ we say the sequence $\{\rho_n\}$ *converges strongly.*

Jørgensen first observed examples of algebraically convergent sequences

$$\rho_n : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{C})$$

of representations whose geometric limit $\Gamma_G = \lim_{n\to\infty}\{\rho_n(\mathbb{Z})\}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$: Γ_G contains $\rho_\infty(\mathbb{Z})$ as a *proper* subgroup and thus ρ_n is a sequence of discrete faithful representations that does *not* converge strongly. Such an example appears in [**Th1**, Ch. 9].

Hyperbolic 3-manifolds. This paper investigates examples that arise when $G = \pi_1(S)$ for S a closed oriented surface. The action of $\rho_n(G)$ by isometries of hyperbolic 3-space determines a quotient hyperbolic 3-manifold $M_n = \mathbb{H}^3/\rho_n(G)$ with the homotopy type of S (in fact $M_n \cong S \times \mathbb{R}$ by a theorem of Thurston and Bonahon [**Th1**, Ch. 9] [**Bon1**]).

Geometric convergence of infinite cyclic subgroups $\langle \rho_n(g)\rangle$, $g \in G$, to Kleinian groups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ was known to arise in this context by Hejhal, Earle, and Marden, and developed explicitly by Kerckhoff and Thurston in [**KT**] and by Thurston in [**Th4**]. Geometrically, the quotients M_n develop a new "torus-end" in the geometric limit hyperbolic manifold $M_G = \mathbb{H}^3/\Gamma_G$ (i.e. M_G has an end homeomorphic to $T \times \mathbb{R}_+$). In this paper, we will first develop this example completely and go on to describe new examples wherein the geometric limit can develop new ends that are products $R \times \mathbb{R}_+$ with surfaces R of higher genus.

To state these examples, we review some terminology.

Definitions. A Kleinian group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Alternately, one may regard it as a discrete subgroup of $\mathrm{Isom}^+ \mathbb{H}^3$, the orientation-preserving isometries of hyperbolic 3-space. There are various objects naturally associated to Γ that come into play.

1. Its *limit set* Λ is the intersection $\overline{\Gamma(0)} \cap \widehat{\mathbb{C}}$ of the orbit of $0 \in \mathbb{H}^3$ with the sphere at infinity.
2. Its *domain of discontinuity* Ω is the complement $\widehat{\mathbb{C}} - \Lambda$. The action of Γ is properly discontinuous on Ω.
3. The *convex hull* $\mathrm{ch}(\Lambda)$ is smallest hyperbolically convex subset of $\mathbb{H}^3$ containing Λ in its closure.

When Γ is torsion-free, the quotient $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic manifold.

In general, we will assume our Kleinian groups to be *non-elementary*: the limit set is required to consist of at least 3 points, which implies that it is in fact uncountable.

By letting the non-elementary group Γ act on $\mathbb{H}^3 \cup \Omega$ we may prolong M to its *Kleinian manifold*

$$\overline{M} = \mathbb{H}^3 \cup \Omega/\Gamma$$

with its natural *conformal boundary* Ω/Γ. Each component of Ω comes equipped with a natural hyperbolic Poincaré metric, and when Γ is finitely generated, Ω/Γ is a collection of Riemann surfaces of finite total area by Ahlfors' finiteness theorem [**Ah**].

A fruitful setting in which to investigate algebraic and geometric convergence is that of Kleinian groups isomorphic to the fundamental group of a closed surface S. The key players are as follows:

Teich(S): The *Teichmüller space* $\mathrm{Teich}(S)$ is the set of equivalence classes of pairs (f, X) of homeomorphisms

$$f\colon S \to X$$

of S to Riemann surfaces X under the equivalence relation $(f, X) \sim (g, Y)$ if there is a conformal map $\phi\colon X \to Y$ such that $\phi \circ f$ is homotopic to g. When S has boundary, f is required to map an annular neighborhood of each boundary component to a neighborhood of a puncture on X.

Mod(S): The *mapping class group* $\mathrm{Mod}(S)$ is the group isotopy classes of orientation preserving homeomorphisms of S. There is a natural action of $\psi \in \mathrm{Mod}(S)$ on $\mathrm{Teich}(S)$ by

$$\psi(f, X) = (f \circ \psi^{-1}, X).$$

***QF*(S):** A *Quasi-Fuchsian group* is a Kleinian group Γ isomorphic to $\pi_1(S)$ stabilizing a directed Jordan curve in $\widehat{\mathbb{C}}$ as its limit set Λ. Any such group determines a pair of Riemann surfaces X and Y homeomorphic to S as the quotient $(\widehat{\mathbb{C}} - \Lambda)/\Gamma$. Equipped with a fixed marking isomorphism $\rho\colon \pi_1(S) \to \Gamma$, X and Y determine a pair $(X, Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$. Conversely, Bers exhibited a bijection

$$Q\colon\ \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S}) \to QF(S)$$

to such *marked* quasi-Fuchsian groups $QF(S)$ modulo conjugacy in $\mathrm{PSL}_2(\mathbb{C})$; any pair of Riemann surfaces is realized by a unique such. We denote by $\Gamma(X, Y)$ the marked quasi-Fuchsian group (up to conjugacy) and reserve $Q(X, Y)$ to refer to the quotient hyperbolic 3-manifold $\mathbb{H}^3/\Gamma(X,Y)$ together with the marking on its fundamental group determined by $\Gamma(X,Y)$.

***AH*(*S*):** We denote by $H(S)$ the set of hyperbolic 3-manifolds M equipped with homotopy equivalences $(f\colon S \to M)$ with the equivalence relation

$$(f\colon S \to M) \sim (g\colon S \to N)$$

if there is an orientation-preserving isometry $\phi\colon M \to N$ such that $\phi\circ f$ is homotopic to g. The set $H(S)$ is identified with discrete faithful representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ modulo conjugacy in $\mathrm{PSL}_2(\mathbb{C})$ by the correspondence $f \mapsto f_*$, and we use these representations to give $H(S)$ the algebraic topology: a sequence of conjugacy classes $\{[\rho_n]\}_{n=1}^{\infty}$ converges to $[\rho_\infty]$ if there are representatives $\rho_n \in [\rho_n]$ and a representative $\rho_\infty \in [\rho_\infty]$ such that for each $g \in \pi_1(S)$, $\rho_n(g)$ converges in $\mathrm{PSL}_2(\mathbb{C})$ to $\rho_\infty(g)$. Denote by $AH(S)$ the set $H(S)$ with the algebraic topology. The space $QF(S)$ sits in $AH(S)$ as an open subset, and the bijection $Q\colon\ \mathrm{Teich}(S)\times\mathrm{Teich}(\overline{S}) \to QF(S)$ is a homeomorphism if we give $QF(S)$ the subspace topology.

When $\partial S \neq \emptyset$ the right boundary condition is that $\rho(\gamma)$ be parabolic for any $\gamma \subset \partial S$. We will often refer to hyperbolic 3-manifolds $M \in AH(S)$ suppressing an implicit marking $f\colon S \to M$.

The modular group also acts on $AH(S)$ by

$$\psi(f\colon S \to M) = (f\circ\psi^{-1}\colon S \to M)$$

for $\psi \in \mathrm{Mod}(S)$.

The Bers slice B_Y: Fixing a Riemann surface Y in the second factor, one obtains a *Bers Slice*

$$B_Y = Q(\mathrm{Teich}(S), Y),$$

an embedded copy of $\mathrm{Teich}(S)$ in the quasi-Fuchsian groups. This *Bers embedding* $b_Y\colon\ \mathrm{Teich}(S) \to QF(S)$ of Teichmüller space given by

$$b_Y(X) = Q(X,Y)$$

depends as the virtue of being precompact in $AH(S)$ for each Y. The resulting compactification $\overline{B_Y}$, however, depends on the Riemann surface Y ([**KT**]).

Bers' boundary: The resulting boundary $\partial B_Y \subset AH(S)$ lies central to many issues in the deformation theory of hyperbolic 3-manifolds. Bers gave a classification of the marked hyperbolic manifolds $(f\colon S \to M) \in \partial B_Y$ in terms of their associated Kleinian manifolds $\overline{M}$. There are three possibilities: either,

1. *M is a cusp:* If there is an element $\gamma \in \pi_1(S)$, $\gamma \not\simeq \partial S$, such that $f_*(\gamma)$ is parabolic then γ is called an *accidental parabolic*, and M is called a *cusp*; or
2. *M is totally degenerate:* If $\partial M = Y$, then M is said to be *totally degenerate*; or
3. *M is partially degenerate:* If the total Poincaré area of ∂M satisfies

$$2\pi\chi(S) < \mathrm{area}(\partial M) < 4\pi\chi(S),$$

then M is *partially degenerate*.

In each case, the uniformizing Kleinian group Γ goes by the same name: Γ is either a cusp, totally degenerate or partially degenerate. It follows from Ahlfors' finiteness theorem that any partially degenerate group must contain an accidental parabolic and therefore be a cusp.

Iteration of mapping classes on a Bers' slice: Choose an element φ of $\mathrm{Mod}(S)$ of infinite order. Then the sequence $\{\varphi^n(X)\}$ tends to ∞ in $\mathrm{Teich}(S)$. How such sequences accumulate on the boundary ∂B_Y under the embedding

$$b_Y(\varphi^n(X)) = Q(\varphi^n(X), Y)$$

was originally investigated in [**Bers4**], and many interesting studies have since followed (see for example [**Shiga**], [**CT**], [**KT**], and [**Mc2**]). In [**Br**] we show that every mapping class has a power whose iteration on any Bers slice converges.

THEOREM 1.1 (Main theorem of [**Br**]). *Given $\varphi \in \mathrm{Mod}(S)$, there is a k such that for each X and Y, the sequence $\{Q(\varphi^{kn}X, Y)\}_{n=1}^{\infty}$ converges algebraically and geometrically.*

Note that the theorem is obvious when φ has finite order, and that the full sequence $\{Q(\varphi^n X, Y)\}_{n=1}^{\infty}$ has finitely many accumulation points.

The proof relies on a geometric analysis of the sequence of hyperbolic manifolds $\{Q(\varphi^{kn}X, Y)\}_{n=1}^{\infty}$. In particular,we construct a precise isometric model for the geometric limit of $\{Q(\varphi^{kn}X, Y)\}$ in the course of the proof. Convergence was established in the special case of *pseudo-Anosov* mapping classes by Cannon and Thurston [**CT**]; the examples we present in this paper are the prototypes for the general case.

Much of the complexity present in the general case arises when S is a closed surface of genus 2 as in figure 2. Here are the two examples we present.

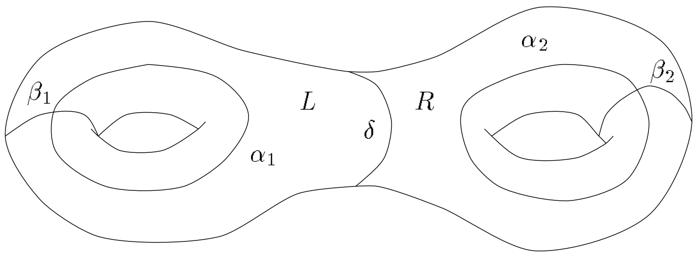

Figure 2. A two-holed torus.

Example 1: Let τ be a *Dehn twist* about the curve $\delta \subset S$ that separates S into two punctured tori L and R. The mapping class τ is the isotopy class of the homeomorphism obtained by cutting S along the curve δ, performing a 2π twist, and re-gluing. The iteration $\{Q(\tau^n X, Y)\}$ converges to a cusp $Q_Y \in \partial B_Y$. If $\rho_n \to \rho_\infty$ are convergent representatives of the corresponding convergent conjugacy classes $[\rho_n] \to [\rho_\infty]$, the geometric limit Γ_G of the Kleinian groups $\{\rho_n(\pi_1(S))\}$ has hyperbolic hyperbolic quotient $M_G = \mathbb{H}^3/\Gamma_G$ with homeomorphism type

$$M_G \cong S \times I^\circ - (\delta, 1/2).$$

Here, I is the closed interval $[0, 1]$ and I° is its interior $(0, 1)$. This example, developed in detail in [**KT**, §3] is discussed in section 3.

Example 2: Let φ be the mapping class given by a product of Dehn twists τ_1 and τ_2 about curves α_2 and β_2. We call φ the *half-pseudo-Anosov* mapping class. The iteration $Q(\varphi^n X, Y)$ converges to a *partially degenerate* algebraic limit Q_Y. Now the geometric limit Γ_G has hyperbolic quotient M_G with homeomorphism type

$$M_G \cong S \times I^\circ - (R, 1/2)$$

where R is the punctured torus subsurface in $S - \delta$ containing α_2 and β_2 as in figure 2.

Plan of the paper. We first discuss preliminaries. In section 3 we discuss example 1 above, and construct the algebraic and geometric limits. In section 4 we develop an important connection between the dynamics of elements of the modular group and quasi-isometry invariants of hyperbolic 3-manifolds arising as limits of their iteration on a Bers slice; the connection lies in the notion of a *measured lamination* on a hyperbolic surface. Section 5 discusses example 2 and its algebraic and geometric limits. Finally, section 6 develops a method for obtaining generators in $\mathrm{PSL}_2(\mathbb{C})$ for the algebraic and geometric limits of iteration of the half-pseudo-Anosov example φ on a certain Bers slice. This method employs a computer algorithm of Curt McMullen ([**Mc2**]) developed to obtain explicit degenerate punctured-torus groups, together with *Klein-Maskit combination.* Computer generated renderings of the limit sets for the algebraic and geometric limits are constructed from this data using a limit set drawing program also written by McMullen[1].

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2. Preliminaries

We review some of the basic tools necessary for the understanding and construction of algebraic and geometric limits of Kleinian groups.

2.1. Hyperbolic 3-manifolds. Given a complete hyperbolic 3-manifold M presented as the quotient $M = \mathbb{H}^3/\Gamma$ of $\mathbb{H}^3$ by a finitely generated torsion free Kleinian group Γ, its *convex core* $\mathrm{core}(M) = \mathrm{ch}(\Lambda)/\Gamma$ is the minimal geodesically convex subset of M. The convex core records much of the interesting geometric information about the manifold M. We say Γ is *geometrically finite* if the volume of an ϵ-neighborhood $\mathcal{N}_\epsilon(\mathrm{core}(M))$ of the convex core has finite volume. Otherwise Γ is geometrically infinite.

A hyperbolic 3-manifold M admits a decomposition into pieces.

DECOMPOSITION 2.1. *A complete hyperbolic 3-manifold M can be decomposed into the following pieces.*

1. **The cuspidal thin part P:** *P is the union of open horoball neighborhoods of cusps of M. The submanifold $M - P \subset M$ is called the* pared submanifold *of M.*
2. **The relative compact core $(\mathcal{M}, \mathcal{P})$:** *A compact submanifold $\mathcal{M}$ with a collection $\mathcal{P}$ of annuli and tori in $\partial\mathcal{M}$ together with an inclusion $\iota\colon (\mathcal{M}, \mathcal{P}) \to (M - P, \partial P)$ such that*
   (a) *ι is a homotopy equivalence,*

[1] The program sources and many examples of limit sets for Kleinian groups are available at the URL: `http://math.berkeley.edu/~ctm/programs.html`.

(b) *for each component P_0 of P, there is a component of $\mathcal{P}$ mapping by ι to P_0.*

3. **The ends $\boldsymbol{E}$:** *a finite collection of* ends E *each bounded by a surface* $S \subset \partial\mathcal{M} - \mathcal{P}$.[2]

An end E is *geometrically finite* when it has finite volume intersection with $\operatorname{core}(M)$. In this case, the nearest point retraction mapping from M to $\operatorname{core}(M)$ determines a natural product structure $S \times \mathbb{R}_+$ on E (see [**Th1**, Ch. 8], [**EM**]).

The complete hyperbolic manifold M is geometrically finite if and only if all ends of its pared submanifold $M - P$ are geometrically finite. When a component $S \subset \partial\mathcal{M} - \mathcal{P}$ is incompressible, meaning $\iota_*|_{\pi_1(S)}$ is injective, we say the end E of $M - P$ it cuts off is *incompressible*, and when each component of $\partial\mathcal{M} - \mathcal{P}$ is incompressible we will say that the pared submanifold has *incompressible ends.*

Action on the sphere at infinity. Let $M = \mathbb{H}^3/\Gamma$ be a complete hyperbolic 3-manifold with finitely generated Kleinian uniformization Γ. By a theorem of Marden [**Mar**], each Riemann surface $X \subset \partial M$ may be adjoined at infinity to compactify a geometrically finite end E of the pared submanifold $M - P$, and each such end determines a finite area surface in ∂M. The surface X is isotopic into a component X_h of the boundary of the convex core $\partial \operatorname{core}(M)$.

For hyperbolic manifolds $(f\colon S \to M) \in AH(S)$ such that $M \in \overline{B_Y}$, the uniformizing Kleinian group $\Gamma = f_*(\pi_1(S))$ stabilizes a simply connected component Ω_Y in its domain of discontinuity and $\Omega_Y / f_*(\pi_1(S)) = Y$. Under these hypotheses we have the following due to Marden (see [**Mar**, Prop. 5.4]).

THEOREM 2.2. *Let $M \in \overline{B_Y}$. Let simple closed curves $\{\gamma_j\}_{j=1}^p$ be its accidental parabolics. Then decomposition 2.1 determines a relative compact core $\iota\colon (S \times I, \mathcal{P}) \to M - P$, where*

$$\mathcal{P} = (\cup_{j=1}^p \mathcal{P}_j) \cup \partial S \times I$$

is a collection of annuli such that $\mathcal{P}_j \subset S \times \{1\}$ has core curve $(\gamma_j, 1) \subset S \times I$. There are surfaces $S_Y \cup \{S_m\}_{m=1}^q$ in $\partial(S \times I) - \mathcal{P}$ cutting off ends E_Y and $\{E_m\}_{m=1}^q$ of the pared submanifold such that

1. *S_Y cuts off a geometrically finite end E_Y, and the image under inclusion $\iota_*(\pi_1(S_Y)) = f_*(\pi_1(S))$ stabilizes a unique invariant component Ω_Y, with $\Omega_Y / f_*(\pi_1(S)) \cong Y$,*
2. *each S_m is incompressible,*
3. *S_m cuts off a geometrically finite end E_m if and only if $f_*(\pi_1(S_m)) = \Gamma_m$ is quasi-Fuchsian and some component $\Omega_m \in \Omega$ gives $\Omega_m/\Gamma_m \cong S_m$,*
4. *S_m cuts off a geometrically infinite end E_m if and only if $f_*(\pi_1(S_m)) = \Gamma_m$ is a totally degenerate group and $\Omega/\Gamma_m \cong S_m$.*

2.2. Tameness. A geometrically infinite end E is said to be *tame* or *simply degenerate* if $E \cong S \times \mathbb{R}_+$. When all ends of the pared submanifold $M - P$ are geometrically finite or tame, the manifold M and its uniformizing Kleinian group Γ are said to be *geometrically tame.* An important result of Thurston and Bonahon guarantees that this tameness holds for all hyperbolic M such that $\pi_1(M)$ is finitely generated and $M - P$ has incompressible ends [**Bon1**] [**Th1**]. All manifolds in $AH(S)$ enjoy this property.

[2] As is customary [**Th1**] [**Bon1**], we abuse notation and use the term "end" to refer to a neighborhood of an end of the topological space $M - P$.

THEOREM 2.3 (Tameness). *Let* $Q = (f\colon S \to M) \in AH(S)$. *Then*

$$M \cong \operatorname{int}(S) \times \mathbb{R}.$$

In particular all Kleinian groups satisfying the hypotheses of theorem 2.2, are geometrically tame.

2.3. Combination theorems. It is often useful to understand a given 3-manifold in terms of a decomposition into simpler ones. Given two compact topological 3-manifolds M_1 and M_2 and a homeomorphism $f\colon F_1 \to F_2$ between surfaces $F_1 \subset \partial M_1$ and $F_2 \subset \partial M_2$, the Seifert-Van Kampen [**Mass**, Thm. 2.1] theorem allows us to determine the fundamental group of the 3-manifold $M = M_1 \cup_{f(F_1)=F_2} M_2$ obtained as a gluing of M_1 and M_2 by f:

$$\pi_1(M) = \pi_1(M_1) *_{\pi_1(F_1)} \pi_1(M_1).$$

The group $\pi_1(M)$ is the *free product with amalgamation* of $\pi_1(M_1)$ and $\pi_1(M_2)$ over the subgroup $\pi_1(F_1) \cong \pi_1(F_2)$.

When $\overline{M_1}$ and $\overline{M_2}$ are Kleinian manifolds, the homeomorphism $f\colon F_1 \to F_2$ between *conformal* boundary components $F_1 \subset \partial M_1$ and $F_2 \subset \partial M_2$ determines a *gluing problem*: when the ends of M_1 and M_2 corresponding to F_1 and F_2 have a compatible shape, then the manifolds M_1 and M_2 can be isometrically glued together to form a new complete hyperbolic manifolds M_f so that $\overline{M_f} \cong \overline{M_1} \cup_f \overline{M_2}$ (see [**Mor**] [**Mc1**]). Then $M_1 \sqcup M_2$ is called a solution to the gluing problem determined by f.

The combination theory of Klein and Maskit allows us to give criteria for solving this problem on the level of Kleinian groups. If Γ_1 and Γ_2 are torsion-free Kleinian groups with a common quasi-Fuchsian subgroup Γ_F that stabilizes components $\Omega_1 \subset \Omega_{\Gamma_1}$ and $\Omega_2 \subset \Omega_{\Gamma_2}$ of the domain of discontinuity for Γ_1 and Γ_2, each Kleinian manifold $\overline{M_1}$ and $\overline{M_2}$ will have a surface in its conformal boundary homeomorphic to the surfaces uniformized by Γ_F. The following gives conditions in which we may "glue" two Kleinian groups along these "peripheral" quasi-Fuchsian subgroups.

THEOREM 2.4 (Klein-Maskit combination I). *Let torsion free Kleinian groups* Γ_1 *and* Γ_2 *have a common quasi-Fuchsian subgroup* Γ_F *that stabilizes components* $\Omega_1 \subset \Omega_{\Gamma_1}$ *and* $\Omega_2 \subset \Omega_{\Gamma_2}$. *Let* $F_i = \Omega_i/\Gamma_F$ *be the Riemann surface in* ∂M_i *uniformized by* Γ_F *and let* $\phi\colon F_1 \to F_2$ *be a homeomorphism such that* $\phi_* = id|_{\Gamma_F}$.

If the limit set Λ_{Γ_F} *separates* $\Lambda_{\Gamma_1} \cap \Omega_{\Gamma_F}$ *and* $\Lambda_{\Gamma_2} \cap \Omega_{\Gamma_F}$ *in* $\widehat{\mathbb{C}}$, *then*

1. *The group* $\langle \Gamma_1, \Gamma_2 \rangle$ *is Kleinian,*
2. $\langle \Gamma_1, \Gamma_2 \rangle \cong \Gamma_1 *_{\Gamma_F} \Gamma_2$, *and*
3. *the Kleinian manifold*

$$\overline{M_{\langle \Gamma_1, \Gamma_2 \rangle}} \cong \overline{M_1} \cup_\phi \overline{M_2},$$

and

4. $\partial M_{\langle \Gamma_1, \Gamma_2 \rangle} = (\partial M_1 - F_1) \sqcup (\partial M_2 - F_2)$.

The first combination theorem treats the *separated* case, in which a surface decomposes a 3-manifold into two components. Decomposition along a non-separating surface gives an *HNN decomposition* of the fundamental group:

$$\pi_1(M) \cong \pi_1(M_1)*_\alpha$$

where M is the gluing $M_1 \cup_\phi$ by $\phi\colon F_1 \to F_2$ a homeomorphism between boundary components $i_1\colon F_1 \hookrightarrow \partial M_1$ and $i_2\colon F_2 \hookrightarrow \partial M_1$ such that for $g \in \pi_1(F_1)$,

$$(i_2)_*(\phi_*(g)) = \alpha \circ (i_1)_*(g) \circ \alpha^{-1}$$

in $\pi_1(M)$.

The first combination theorem translates into this case.

THEOREM 2.5 (Klein-Maskit combination II). *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold, and let Γ have quasi-Fuchsian subgroups Γ_1 and Γ_2 that stabilize components Ω_1 and Ω_2 of Ω_Γ. Let $\Omega_i/\Gamma_i = F_i \subset \partial M$, $i = 1, 2$, and let $\Gamma_2 = \beta \cdot \Gamma_1 \cdot \beta^{-1}$ for some element $\beta \in \mathrm{PSL}_2(\mathbb{C})$. Let $\phi\colon F_1 \to F_2$ be a homeomorphism such that $\phi_*(\gamma) = \beta \cdot \gamma \cdot \beta^{-1}$.*

If the limit set Λ_{Γ_1} separates $\Lambda_\Gamma \cap \Omega_{\Gamma_1}$ and $\Lambda_{\beta\Gamma\beta^{-1}} \cap \Omega_{\Gamma_1}$ then

1. *the group $\langle \Gamma, \beta \rangle$ is Kleinian,*
2. $\langle \Gamma, \beta \rangle \cong \Gamma *_\beta$,
3. *the Kleinian manifold $\overline{M_{\langle \Gamma, \beta \rangle}}$ has homeomorphism type $\overline{M_\Gamma} \cup_\phi$, and*
4. $\partial \overline{M_{\langle \Gamma, \beta \rangle}} = \partial M_\Gamma - (F_1 \sqcup F_2)$.

Remarks: Claim (3) of theorems 2.4 and 2.5 for arbitrary torsion-free Kleinian groups are shown in [**Br**, App. B]. No assumptions of tameness or finite generation are necessary. Note that (3) also implies the weaker statement that the pared submanifold of the combination is a gluing of the pared submanifold(s) corresponding to the subgroup(s) along some surfaces S_m and S'_m (as in decomposition 2.1).

2.4. The geometric topology. A hyperbolic 3-manifold M determines a Kleinian group only up to conjugacy in $\mathrm{PSL}_2(\mathbb{C})$. A *baseframe* ω i.e. a basepoint $x \in M$ and an orthonormal frame ω at x is needed to determine a unique Kleinian group Γ such that in the quotient $\mathbb{H}^3/\Gamma$ the standard frame at the origin projects to ω.

Geometric convergence of Kleinian groups determines an equivalent notion on the quotients: a sequence of manifolds with baseframe (M_n, ω_n) converges geometrically to its geometric limit $(M_\infty, \omega_\infty)$ if for any compact set $K \subset M_\infty$ with $\omega_\infty \in K$ there are L-quasi-isometries[3] $f_n\colon K \to M_n$ such that $f_n(\omega_\infty) = \omega_n$ and f_n tends to an isometry in the C^∞ topology.

We record some facts about geometric convergence.

THEOREM 2.6. *Let $\mathcal{H}_{r,R}$ denote the set of hyperbolic 3-manifolds (M, ω) with baseframe $\omega \in \mathrm{core}(M)$ such that the injectivity radius is bounded below by r at ω and bounded above by R throughout $\mathrm{core}(M)$. Then $\mathcal{H}_{r,R}$ is compact in the geometric topology. The limit set Λ_M varies continuously on $\mathcal{H}_{r,R}$ in the Hausdorff topology on closed subsets of $\widehat{\mathbb{C}}$.*

For a proof see [**Mc2**, Prop. 2.4]. We remark that any algebraically convergent sequence $[\rho_n] \to [\rho_\infty]$ in a Bers slice has convergent representatives $\rho_n \to \rho_\infty$ so that the based quotients $(M_n, \omega_n) = (\mathbb{H}^3, \widetilde{\omega})/\rho_n(\pi_1(S))$ range in $\mathcal{H}^3_{r,R}$ ([**Bon1**] [**Can**, Thm. 6.2]) where $\widetilde{\omega} \in \mathbb{H}^3$ is the standard frame at the origin in $\mathbb{H}^3$. Thus we may always extract a geometric limit of an algebraically convergent sequence in $\overline{B_Y}$, after passing to a subsequence.

[3] An L-quasi-isometry is a diffeomorphism such that $1/L \le |Df(v)|/|v| \le L$.

Hausdorff convergence of the limit sets $\{\Lambda_n\}$ is easily verified to be equivalent to *Carathéodory convergence* of the domains of discontinuity $\{\Omega_n\}$ [**KT**, §2] [**JM**]: a sequence $\Omega_n \subset \widehat{\mathbb{C}}$ of open sets converges to an open set $\Omega \subset \widehat{\mathbb{C}}$ in the sense of Carathéodory if

1. given any compact set $K \subset \Omega$, $K \subset \Omega_n$ for all but finitely many n, and
2. if an open set $\Omega_0 \subset \Omega_{n_k}$ for infinitely many n_k, then $\Omega_0 \subset \Omega$.

In Bers' discussion of B_Y (see [**Bers1**]), the quasi-Fuchsian group $\Gamma(X,Y)$ is obtained from a Fuchsian group $\Gamma(\overline{Y},Y)$, via a *simultaneous uniformization*, namely a quasi-conformal conjugacy

$$w\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

such that

$$w\Gamma(\overline{Y},Y)w^{-1} = \Gamma(X,Y).$$

The mapping w is taken to be conformal on the lower half plane $\mathbb{L}$ and determines a group equivariant quasi-conformal deformation of the upper half plane. The limit set of $\Gamma(X,Y)$ is the image of the equator under w. Normalizing w to fix a point $p \in \mathbb{L}$ and have $w'(p) = 1$ determines an identification of a Bers slice with a family of univalent mappings of $\mathbb{L}$ into $\widehat{\mathbb{C}}$ fixing p with derivative 1 at p. Algebraic convergence of a sequence of quasi-Fuchsian representations is equivalent to uniform convergence on compact sets (locally uniform convergence) of the corresponding univalent mappings (see [**Bers2**] [**KT**]). This normalization pins down a specific representation $\rho\colon \pi_1(S) \to \Gamma(X,Y)$ in the conjugacy class $[\rho]$ determined by $Q(X,Y)$; for representations in a Bers slice, we assume unless otherwise specified that such a normalization has been made and refer unambiguously to the representation $\rho \in B_Y$.

While a locally uniformly convergent sequence w_n so normalized has a univalent limit w_∞ ([**LV**] [**Gard**, §1.8, Lem. 6]) satisfying the same normalization, the image $w_\infty(\mathbb{L})$ can disagree with the Carathéodory limit Ω of the domains $\{w_n(\mathbb{L})\}$. The Carathéodory convergence theorem (see [**Du**, pp. 78]), however, guarantees that

$$w_\infty(\mathbb{L}) \subset \Omega.$$

This has the following fortunate consequence (see [**Br**, Cor. 2.12.13]).

Proposition 2.7. *Let $Q_n \in B_Y$ tend to $Q_Y \in \partial B_Y$ and let the corresponding Kleinian groups $\{\rho_n(\pi_1(S))\}$ converge geometrically to a Kleinian group Γ_G with hyperbolic quotient M_G. Then the covering $Q_Y \to M_G$ extends to an embedding on Y.*

We reprise the proof.

Proof: Let $\rho_n : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ be determined by Q_n via the simultaneous uniformizations w_n normalized as above. The algebraic limit $\rho_\infty(\pi_1(S))$ stabilizes the component $w_\infty(\mathbb{L}) = \Omega_Y$ in its domain of discontinuity Ω_∞ with $\Omega_Y/\rho_\infty(\pi_1(S)) = Y$. Since $\rho_\infty(\pi_1(S)) < \Gamma_G$, and $\Omega_Y \subset \Omega_{\Gamma_G}$ by the Carathéodory convergence theorem, it follows that $\rho_\infty(\pi_1(S)) < \mathrm{Stab}_{\Gamma_G}(\Omega_Y)$. Thus there is a covering of finite area Riemann surfaces

$$\Omega_Y/\rho_\infty(\pi_1(S)) \to \Omega_Y/\,\mathrm{Stab}_{\Gamma_G}(\Omega_Y)$$

by the Ahlfors finiteness theorem. Any such cover must be finite sheeted, so $\rho_\infty(\pi_1(S))$ has at most finite index in $\mathrm{Stab}_{\Gamma_G}(\Omega_Y)$. To see that $\mathrm{Stab}_{\Gamma_G}(\Omega_Y) = \rho_\infty(\pi_1(S))$, we recall a familiar argument which appears in **[JM]**.

Were the index $k > 1$, there would be some $\gamma \in \mathrm{Stab}_{\Gamma_G}(\Omega_Y)$ such that $\gamma^k \in \rho_\infty(\pi_1(S))$. Algebraic convergence implies that there is an element g in $\pi_1(S)$ such that $\lim_{n\to\infty} \rho_n(g) = \gamma^k$. Since $\gamma \in \Gamma_G$ there are $g_n \in \pi_1(S)$ such that $\rho_n(g_n) \to \gamma$ in $\mathrm{PSL}_2(\mathbb{C})$. It follows that $\rho_n(g \cdot g_n^{-k}) \to id$ as $n \to \infty$. By discreteness, $\rho_n(g \cdot g_n^{-k}) = id$ for all n sufficiently large. By faithfulness of ρ_n, $g_n^k = g$ for all such n. Since in a surface group $\pi_1(S)$ the element g has a unique k-th root, $g_n = h \in \pi_1(S)$ for large n, which guarantees that $\gamma = \lim_{n\to\infty} \rho_n(h)$ is contained in $\rho_\infty(\pi_1(S))$. Thus the algebraic limit is the full stabilizer of Ω_Y in the geometric limit.

Since $\Omega_Y / \mathrm{Stab}_{\Gamma_G}(\Omega_Y) = Y$, it follows that $Y \subset \partial M_G$. □

3. Iteration of a Dehn twist

In this section we give perhaps the simplest example of a sequence $Q(X_n, Y) = \mathbb{H}^3/\rho_n(\pi_1(S))$ in a Bers slice whose corresponding algebraic limit representation ρ_∞ has image properly contained in geometric limit Γ_G of $\{\rho_n(\pi_1(S))\}$. The example, studied by Earle, Marden, Hejhal, and others, arises from iteration of a single Dehn twist about a simple closed curve. A complete presentation appears in **[KT]**.

3.1. Algebraic and geometric limits. Let S be a genus two Riemann surface as in figure 2. Let $\tau \in \mathrm{Mod}(S)$ be a Dehn twist about the separating curve δ, which divides S into two punctured tori labeled L and R.

Consider the sequence $Q_n = Q(\tau^n X, Y)$ in $\overline{B_Y}$. Extract a convergent subsequence with algebraic limit ρ_∞ determining quotient $Q_Y = \mathbb{H}^3/\rho_\infty(\pi_1(S))$. Let $\Gamma_Y = \rho_\infty(\pi_1(S))$ denote the image. After passing to a further subsequence, the images $\{\rho_n(\pi_1(S))\}$ converge geometrically to a group Γ_G.

Remark: Theorem 1.1 guarantees algebraic and geometric convergence of iteration of τ^k for some k, and in fact for a mapping class such as τ, k may taken to be 1. While this makes passing to subsequences extraneous, we include this step throughout for logical consistency.

The rest of this section will be devoted to a discussion of the following result.

THEOREM 3.1 (Kerckhoff-Thurston). *The geometric limit $M_G = \mathbb{H}^3/\Gamma_G$ has the homeomorphism type*

$$M_G \cong S \times I^\circ - (\delta, 1/2).$$

(See **[KT**, §3]). In other words, the geometric limit M_G is homeomorphic to a product, with the curve δ *drilled* at level $1/2$ in the product structure. It admits a relative compact core given by

$$(\mathcal{M}_G, \mathcal{P}_G) = (S \times I - V_\delta, \partial V_\delta)$$

where V_δ is an open solid torus neighborhood of $(\delta, 1/2)$ in $S \times I$ (see figure 3).

Along the way we will show that the algebraic limit $Q_Y = \mathbb{H}^3/\Gamma_Y$ has associated pared manifold $(\mathcal{M}, \mathcal{P}) = (S \times I, A_\delta)$ where A_δ is an annular neighborhood of $(\delta, 1)$ in $S \times I$.

The construction goes via gluing together the geometric limit from its surface subgroups.

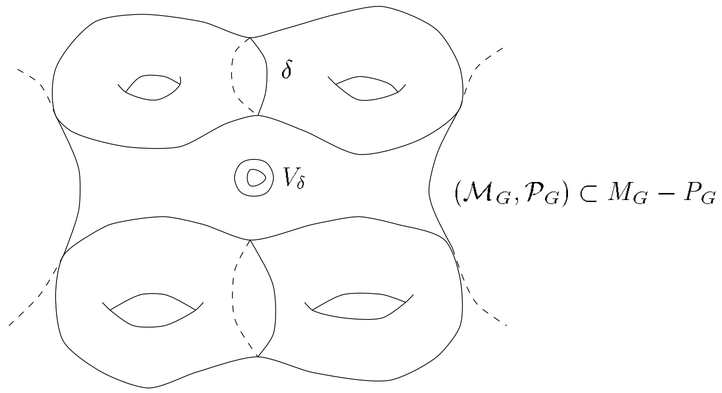

Figure 3. The geometric limit of iteration of a Dehn twist.

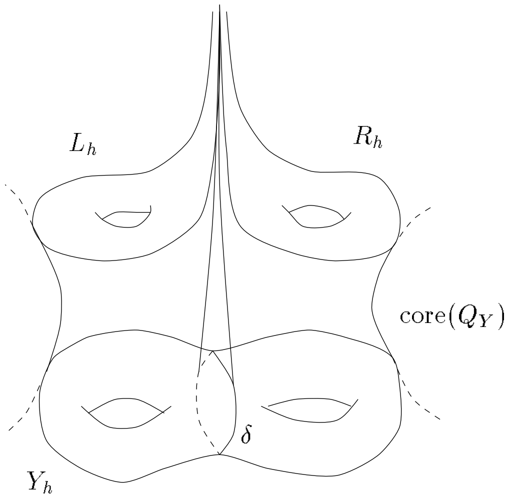

Figure 4. The algebraic limit of iteration of a Dehn twist.

Sketch of proof: We present a slight variant of the discussion in [**KT**].

I. The re-marking trick. To begin with, we "re-mark" the sequence $\{\rho_n\}$ by iterating τ^{-1} on $AH(S)$. The sequence

$$Q_n' = \tau^{-n}(Q_n) = Q(X, \tau^{-n} Y)$$

ranges in a precompact slice $Q(X, \mathrm{Teich}(\overline{S}))$ in $AH(S)$. Pass to a further convergent subsequence and extract a limit Q_X.

The Q_n' determine representations in the conjugacy class of

$$\rho_n' = \rho_n \circ \tau_*^n$$

in $\mathrm{PSL}_2(\mathbb{C})$. Choosing a basepoint x for $\pi_1(S)$ in L, the automorphism τ_* may be chosen so that $\tau_*(\alpha_1) = \alpha_1$ and likewise for β_1, while $\tau_*(\alpha_2) = \delta \cdot \alpha_2 \cdot \delta^{-1}$ and likewise for β_2. The representation ρ_n' differs from ρ_n only by precomposition by an outer automorphism of $\pi_1(S)$ so the sequences $\{\rho_n(\pi_1(S))\}$ and $\{\rho_n'(\pi_1(S))\}$ have the same geometric limit Γ_G.

II. Non-strong convergence: $\Gamma_Y \neq \Gamma_G$. The representations ρ'_n converge up to conjugacy, which means there are elements $\phi_n \in \mathrm{PSL}_2(\mathbb{C})$ such that

$$\phi_n \cdot \rho'_n \cdot \phi_n^{-1}$$

converges algebraically to a representation ρ'_∞. Let $\Gamma_X = \rho'_\infty(\pi_1(S))$ denote its image so that $Q_X = \mathbb{H}^3/\Gamma_X$. Note that since $\rho_n(\alpha_1) = \rho'_n(\alpha_1)$ and $\rho_n(\beta_1) = \rho'_n(\beta_1)$, and ρ_n converges on α_1 and β_1, the elements ϕ_n may be taken to be the identity. It follows that since

$$\rho'_n(\alpha_2) = \rho_n(\delta^n) \cdot \rho_n(\alpha_2) \cdot \rho_n(\delta^{-n})$$

and

$$\rho'_n(\beta_2) = \rho_n(\delta^n) \cdot \rho_n(\beta_2) \cdot \rho_n(\delta^{-n})$$

the elements $\rho_n(\delta^n)$ converge in $\mathrm{PSL}_2(\mathbb{C})$: given a triple η_1, η_2 and η_3 of non-commuting elements of $\pi_1(R)$ each $\rho_n(\delta^n)$ must exchange the three distinct attracting fixed points of their images under ρ_n and ρ'_n, which converge to distinct triples of points in $\widehat{\mathbb{C}}$. Hence $\rho_n(\delta^n)$ converges in $\mathrm{PSL}_2(\mathbb{C})$ as $n \to \infty$. Let $\lim_{n\to\infty} \rho_n(\delta^n) = \overline{\delta}$. Note that $\overline{\delta} \in \Gamma_G$.

Claim: $\overline{\delta} \notin \Gamma_Y$, Γ_X. Otherwise, there would be some $g \in \pi_1(S)$ such that $\rho_n(g) \to \overline{\delta}$. This would imply that $\rho_n(g \cdot \delta^{-n}) \to id$ which, by discreteness, guarantees that $\rho_n(g \cdot \delta^{-n}) = id$ for all sufficiently large n. By faithfulness of ρ_n, $g = \delta^{-n}$ for sufficiently large n, a contradiction. The same applies to ρ'_n.

III. The element $\rho_\infty(\delta)$ is parabolic. Since $\rho_n(\delta)$ commutes with $\rho_n(\delta^n)$ for all n, $\rho_\infty(\delta)$ commutes with $\overline{\delta}$. If $\langle \rho_\infty(\delta), \overline{\delta} \rangle$ were isomorphic to $\mathbb{Z}$, there would be integers j and k such that $\rho_\infty(\delta^j) = \overline{\delta}^k$. This implies that $\rho_n(\delta^{j-kn}) \to id$ as $n \to \infty$ and thus $\rho_n(\delta^{j-kn})$ is the identity for sufficiently large n, contradicting faithfulness of ρ_n. It follows that

$$\langle \rho_\infty(\delta), \overline{\delta} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

and the only possibility for such a subgroup is a rank 2 parabolic subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Hence, $\rho_\infty(\delta)$ and $\overline{\delta}$ are parabolic.

IV. The subgroups $\rho_\infty(\pi_1(L))$ and $\rho_\infty(\pi_1(R))$ are quasi-Fuchsian. To understand the geometry of the quotients $\mathbb{H}^3/\rho_\infty(\pi_1(L))$ and $\mathbb{H}^3/\rho_\infty(\pi_1(R))$ we refer to 3-dimensional techniques developed by Thurston and Bonahon for understanding a hyperbolic 3-manifolds using its closed geodesics. Let $\mathcal{S}$ denote the set of essential homotopy classes of simple closed curves on S. Each homotopy class $\gamma \in \mathcal{S}$ has a unique geodesic representative in any hyperbolic structure on S. The set $\mathcal{S}$ admits a well defined intersection number

$$i \colon \mathcal{S} \times \mathcal{S} \to \mathbb{Z}_+$$

such that $i(\gamma, \gamma) = 0$ and $i(\gamma, \eta)$ counts the minimal number of intersections of curves in each isotopy class.

We have the following from [**Br**, Ch. 4].

Lemma 3.2. *Let $\varphi \in \mathrm{Mod}(S)$ preserve the isotopy class of γ and η such that $i(\gamma, \eta) \neq 0$. Then for any accumulation point $(f \colon S \to Q_Y) \in \partial B_Y$ of $Q(\varphi^n X, Y)$, we have*

1. *neither $f_*(\gamma)$ nor $f_*(\eta)$ is parabolic,*

2. *there is a subsurface* $T \subset S$ *such that* $\gamma \cup \eta \subset T$ *and* $f_*(\pi_1(T))$ *is quasi-Fuchsian.*

Since τ stabilizes the isotopy class of every simple closed curve in L and R, the lemma guarantees that the conformal boundary ∂Q_Y consists of the Riemann surface Y, and two punctured tori L and R uniformized by $\rho_\infty(\pi_1(L)) = \Gamma_L$ and $\rho_\infty(\pi_1(R)) = \Gamma_R$. Likewise, the conformal boundary ∂Q_X consists of the Riemann surface X and punctured tori L' and R' uniformized by the quasi-Fuchsian subgroups $\rho'_\infty(\pi_1(L)) = \Gamma'_L$ and $\rho'_\infty(\pi_1(R)) = \Gamma'_R$.

V. Klein-Maskit combination. We now combine Γ_X and Γ_Y to form the geometric limit. First note that Γ_X and Γ_Y share the quasi-Fuchsian subgroup Γ_L. Moreover, the element $\overline{\delta}$ has the property that

$$\Gamma'_R = \overline{\delta} \cdot \Gamma_R \cdot \overline{\delta}^{-1}.$$

Let Λ_X and Λ_Y denote the limit sets for Γ_X and Γ_Y respectively, and let Ω_G be the domain of discontinuity for the geometric limit. The Carathéodory convergence theorem (see section 2.4, [**Du**, pp. 78]) together with observations of [**KT**, §2] guarantee that Ω_Y and Ω_X each embed disjointly in Ω_G, so since

$$\Lambda_X = \partial\Omega_X \quad \text{and} \quad \Lambda_Y = \partial\Omega_Y$$

and the Jordan curve limit set Λ_L of Γ_L lies in the intersection $\Lambda_X \cap \Lambda_Y$, the limit set Λ_L must separate the intersection $\Lambda_X \cap \Omega_L$ of Λ_X with the domain of discontinuity for Γ_L from the intersection $\Lambda_Y \cap \Omega_L$. Hence the hypotheses of the first combination theorem are satisfied. Thus, letting $Q_X = \mathbb{H}^3/\Gamma_X$ and $Q_Y = \mathbb{H}^3/\Gamma_Y$, we have

1. the group $\langle \Gamma_Y, \Gamma_X \rangle$ is Kleinian,
2. $\langle \Gamma_Y, \Gamma_X \rangle \cong \Gamma_Y *_{\Gamma_L} \Gamma_X$, and
3. the associated Kleinian manifold $\overline{\mathbb{H}^3/\langle \Gamma_Y, \Gamma_X \rangle}$ has the homeomorphism type $\overline{Q_Y} \cup_\phi \overline{Q_X}$, the gluing along the punctured tori $L \subset \partial Q_Y$ and $L' \subset \partial Q_X$ by a homeomorphism ϕ such that $\phi_* = id|_{\Gamma_L}$.

Denote by $\Gamma_c = \langle \Gamma_X, \Gamma_Y \rangle$ this first combination along Γ_L.

By similar reasoning, the limit set Λ_R of Γ_R must separate $\Lambda_{\Gamma_c} \cap \Omega_R$ from $\Lambda_{\overline{\delta}\Gamma_c\overline{\delta}^{-1}} \cap \Omega_R$, so the second combination theorem applies, and we have

$$\langle \Gamma_c, \overline{\delta} \rangle \cong (\Gamma_Y *_{\Gamma_L} \Gamma_X)*_{\overline{\delta}}$$

(note that $\langle \Gamma_c, \overline{\delta} \rangle = \langle \Gamma_Y, \overline{\delta} \rangle$).

The resulting group which we denote $\Gamma_B = \langle \Gamma_c, \overline{\delta} \rangle$, is a subgroup of the geometric limit. Let $M_B = \mathbb{H}^3/\Gamma_B$ denote its quotient. Then we have:

1. the Kleinian manifold $\overline{M_B} \cong S \times [0,1] - (\delta, 1/2)$, and
2. the conformal boundary ∂M_B consists of Riemann surfaces X and Y and the natural covering projection $M_B \to M_G$ extends to an embedding on $X \sqcup Y = \partial M_B$.

It follows that this covering must be a homeomorphism, so $\Gamma_B = \Gamma_G$. This completes the construction. □

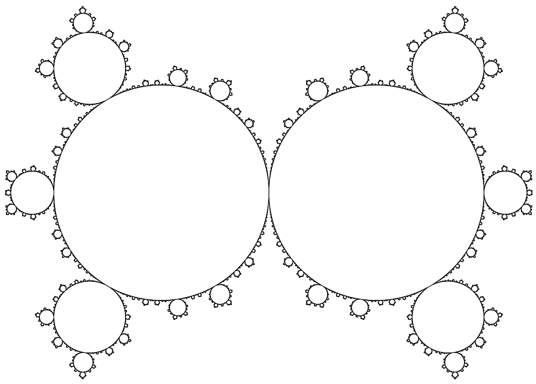

Figure 5. The limit set for the algebraic limit Γ_Y of iteration of a Dehn twist.

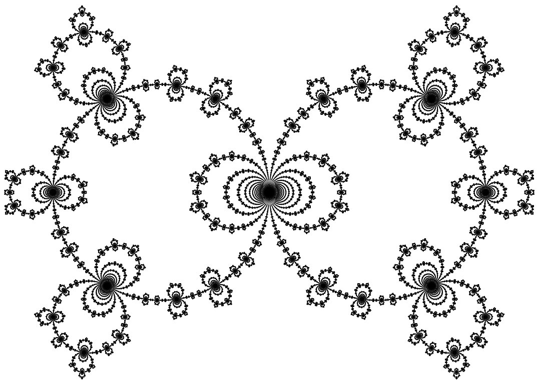

Figure 6. The limit set of the geometric limit Γ_G.

3.2. Limit sets. The construction of the geometric limit from algebraic limit subgroups has the virtue of giving an algorithm for finding explicit generators in $\mathrm{PSL}_2(\mathbb{C})$ for the groups Γ_Y and Γ_G.

Such an explicit description is used by Kerckhoff and Thurston in their proof of the non-continuity of the action of the modular group on a Bers boundary of Teichmüller space [**KT**, Thm. 2]. While for the group they construct the subgroups $\rho_\infty(\pi_1(L))$ and $\rho_\infty(\pi_1(R))$ are non-Fuchsian quasi-Fuchsian subgroups, the case when they are Fuchsian exhibits the symmetry of figures 5 and 6.

We will apply these techniques to the instance of iteration of a *partially pseudo-Anosov* mapping class, e.g. the product of Dehn twists about α_2 and β_2 described in example 2, in section 5.

4. Laminations

The analysis of iteration of a pseudo-Anosov mapping class on a Bers slice is enabled by making use of *measured laminations*. These are points in the closure of the embedding

$$\iota \colon \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}}$$

given by

$$\iota(t, \alpha) = \langle t \cdot i(\alpha, \beta) \ : \ \beta \in \mathcal{S} \rangle.$$

We denote this completion of $\mathbb{R}_+ \times \mathcal{S}$ by $\mathcal{ML}(S)$. The space $\mathcal{ML}(S)$ is homeomorphic to the Euclidean space of dimension $6g-6$ where g is the genus of S (see [**Bon2**] for an interesting discussion of laminations and related *geodesic currents*). Thurston showed that the projectivization $\mathcal{PL}(S)$ of the non-zero elements of this space by scaling of measures, or weights, gives a natural compactification of the Teichmüller space such that $\mathrm{Teich}(S) \cup \mathcal{PL}(S) \cong D^{6g-6}$, the closed unit ball in $\mathbb{R}^{6g-6}$ [**Th3**] [**FLP**].

The modular group extends continuously to $\partial\, \mathrm{Teich}(S) = \mathcal{PL}(S)$. In [**Th3**], Thurston uses the Brouwer fixed point theorem to classify elements φ of $\mathrm{Mod}(S)$ in terms of their action on $\mathcal{PL}(S)$. Let S be endowed with a hyperbolic structure for reference.

Theorem 4.1 (Thurston). *Let $\varphi \in \mathrm{Mod}(S)$. Then either*

1. **φ has finite order:** *there is some $X \in \mathrm{Teich}(S)$ such that $\varphi(X) = X$,*
2. **φ is pseudo-Anosov:** *there is a pair of measured laminations μ^u and $\mu^s \in \mathcal{ML}(S)$ and $c > 1$ such that $\varphi(\mu^u) = c\mu^u$ and $\varphi(\mu^s) = \frac{1}{c}\mu^s$.*
3. **φ is reducible:** *φ permutes some collection of pairwise disjoint simple closed geodesics $\{\gamma_1, \dots, \gamma_n\}$ up to isotopy, that decompose the surface S into surfaces on which φ satisfies either (1) or (2).*

The laminations μ^s and μ^u, called the stable and unstable laminations for φ, are uniquely determined up to scale (see [**Th3**] [**FLP**]).

When a reducible mapping class φ restricts to a pseudo-Anosov mapping class on a subsurface, we will call it a *partially pseudo-Anosov* mapping class (Bers calls these mapping classes *pseudo-hyperbolic* [**Bers3**]). This partitions the reducible classes into those consisting of commuting Dehn twists about disjoint simple closed curves and those with some pseudo-Anosov behavior.

Among the useful invariants for the a hyperbolic structure $X \in \mathrm{Teich}(S)$ are the lengths of simple closed curves on the surface. The geodesic length function

$$\mathrm{length} \colon \mathcal{S} \times \mathrm{Teich}(S) \to \mathbb{R}_+$$

assigns to each pair (γ, X) of an isotopy class of simple closed curves $\gamma \in \mathcal{S}$ and a Riemann surface $X \in \mathrm{Teich}(S)$ the length of the geodesic representative of the isotopy class in the Poincaré metric on X. After extending by linearity to weighted simple closed curves $\mathcal{S} \times \mathbb{R}_+$, there is a continuous extension to $\mathcal{ML}(S)$

$$\mathrm{length} \colon \mathcal{ML}(S) \times \mathrm{Teich}(S) \to \mathbb{R}_{\geq 0}$$

(see [**Bon1**, Prop. 4.1]).

For a quasi-Fuchsian manifold $Q(X, Y)$ one can associate to a given curve $\gamma \in \mathcal{S}$ the length of the unique geodesic γ^* in the free homotopy class of γ in $Q(X, Y)$. We say γ is *realized* in the 3-manifold by the geodesic γ^*.

While there is no extension of the hyperbolic metric on $Q(X, Y)$ to a metric on the conformal boundary, there is nevertheless a bound on the lengths of geodesics γ^*, $\gamma \in \mathcal{S}$ in terms of the Poincaré metrics on X and Y.

THEOREM 4.2 (Bers). *Let X, $Y \in \mathrm{Teich}(S)$ and let $\gamma \subset S$ be a simple closed curve.*

$$\mathrm{length}_{Q(X,Y)}(\gamma) \leq 2 \min\{\mathrm{length}_X(\gamma), \mathrm{length}_Y(\gamma)\}.$$

(See [**Bers2**, Thm. 3] [**Mc1**, Prop. 6.4]).

A *geodesic lamination* on a finite area hyperbolic surface X is a closed subset $\lambda \subset X$ given as a union of disjoint simple complete geodesics called *leaves* of the lamination. To any measured lamination μ, one may associate its *support* $\mathrm{supp}(\mu)$, which is a geodesic lamination (see [**CEG**, §4] [**Bon2**]).

Pleated surfaces. A general geodesic lamination can also be realized in $Q(X, Y)$ via the notion of a *pleated surface*. A pleated surface is a pair (f, T) of a hyperbolic surface T together with a *path isometric* map $f: T \to Q(X, Y)$ (f sends rectifiable arcs in X to rectifiable arcs in $Q(X, Y)$ of the same length) such that for any $x \in T$, f maps a geodesic segment through x isometrically to a geodesic segment through $f(x)$. The map f "pleats" the surface along some geodesic lamination λ ([**Th2**, Prop. 5.9]), and is totally geodesic in the complement of λ. The lamination λ is called the *pleating locus* of the pleated surface (f, T).

Given a pleated surface $f: T \to M$, we say a geodesic lamination $\lambda_0 \subset T$ is *realizable* in the homotopy class of f if its intersection with the pleating locus of (f, T) is again a geodesic lamination. Each leaf of λ_0 is mapped isometrically by f to its image (see [**Th1**, ch. 8] [**CEG**, §5]). In a quasi-Fuchsian manifold $(f: S \to Q(X, Y))$ every geodesic lamination is *realizable* in the homotopy class of f.

When the pleating locus λ contains a simple closed curve γ, the length of $\gamma^* \subset Q(X, Y)$ is the same as its length in the hyperbolic surface T. In this way we may define another length function

$$\mathrm{length}: \mathcal{S} \times QF(S) \to \mathbb{R}_+.$$

Given $M \in AH(S)$, we may measure the lengths of realizable simple closed curves with weights simply by taking the length in the realizing pleated surface. Likewise, when the support of a measured lamination $\mathrm{supp}(\mu)$ is realized by a pleated surface (f, T), we may define

$$\mathrm{length}_M(\mu) = \mathrm{length}_T(\mu).$$

Measuring the length of a measured lamination in this way extends the above length function to all of $\mathcal{ML}(S) \times QF(S)$.

For general marked hyperbolic 3-manifolds $(f: S \to M) \in AH(S)$ a theorem of Thurston [**Th4**] (whose proof has yet to appear in print) guarantees that the length of a measured lamination is a continuous function on the set $\mathcal{R} \subset \mathcal{ML}(S) \times AH(S)$ consisting of pairs (μ, M) where $\mathrm{supp}(\mu)$ is realizable in the hyperbolic 3-manifold M by a pleated surface in the homotopy class of f.

Non-realizable laminations: Let $(f\colon S \to M)$ belong to $AH(S)$. When a non-peripheral element $\gamma \in \pi_1(S)$ has parabolic image $f_*(\gamma)$, there is no geodesic representative in the quotient manifold $\mathbb{H}^3/\rho(\pi_1(S))$. As a geodesic lamination, γ is said to be *non-realizable*. A general geodesic lamination λ is said to be *non-realizable* when there is no pleated surface in the homotopy class of f realizing the lamination. Non-realizable laminations can arise only when $(f\colon S \to M) \in AH(S)$ is not quasi-Fuchsian.

If a lamination is realizable in $M \in AH(S)$, then it has positive length. Likewise, in [**Br**] we show the following:

THEOREM 4.3. *Let $\{(\mu_n, M_n)\}$ be a sequence of pairs belonging to $\mathcal{R}$ such that $\mu_n \to \mu$ in $\mathcal{ML}(S)$ and $M_n \to M$ in $AH(S)$. Then if*

$$\lim_{n\to\infty} \mathrm{length}_{M_n}(\mu_n) = 0$$

the support $\mathrm{supp}(\mu)$ *is non-realizable in M.*

For M in a Bers compactification ∂B_Y, non-realizable non-peripheral simple closed curves are precisely the accidental parabolics of M. Likewise,

* *when there are* no *accidental parabolics, there exists a non-realizable lamination if and only if M is totally degenerate*

(see [**Th1**, §9] [**Br**]). Thus the above theorem gives useful criteria for determining when a hyperbolic manifold in a Bers slice is totally degenerate.

5. Iteration

We first consider iteration of a pseudo-Anosov mapping class ψ on a surface S of genus 2. This case is now well understood by work of Thurston, Cannon, and McMullen (see [**CT**] and [**Mc2**]).

5.1. Iteration of a pseudo-Anosov class. Consider the sequence $Q_n = Q(\psi^n X, Y)$ ranging in the Bers slice B_Y. Let μ^u be the unstable lamination for ψ. Then by theorem 4.2 we have

$$\begin{aligned}
\mathrm{length}_{Q_n}(\mu^u) &\leq K\,\mathrm{length}_{\psi^n X}(\mu^u) \\
&= K\,\mathrm{length}_X(\psi^{-n}(\mu^u)) \\
&= K\,\mathrm{length}_X(\frac{1}{c^n}\mu^u) \\
&= \frac{K}{c^n}\,\mathrm{length}_X(\mu_u)
\end{aligned}$$

where $c > 1$. It follows that

$$\mathrm{length}_{Q_n}(\mu^u) \to 0$$

as $n \to \infty$, so for any accumulation point Q_Y of $\{Q_n\}$, the support of the unstable lamination $\mathrm{supp}(\mu^u)$ is non-realizable in Q_Y.

An important observation due to Bers ([**Bers4**] see also [**Mc2**, §3]) is that any limit $(f\colon S \to Q_Y)$ of iteration of a mapping class $\psi \in \mathrm{Mod}(S)$ on a Bers slice admits a quasi-isometry $\phi\colon Q_Y \to Q_Y$ in the homotopy class of ψ: i.e. ϕ satisfies $f \circ \psi \simeq \phi \circ f$. The mapping class is said to *quasi-isometrically realized* on Q_Y. Since any self-quasi-isometry of Q_Y must preserve the collection of parabolic isotopy classes in $\mathcal{S}$, and ψ preserves *no* collection of isotopy classes in $\mathcal{S}$, it follows that the limit Q_Y has no accidental parabolics (see [**Mc2**, Thm. 3.7]).

Thus, since $\mathrm{supp}(\mu^u)$ is non-realizable in Q_Y and Q_Y has no accidental parabolics, it follows that Q_Y is totally degenerate.

5.2. Iteration of a partially pseudo-Anosov class. We now present the remaining building block for understanding iteration of a general element in $\mathrm{Mod}(S)$ on a Bers slice. Again let S be a surface of genus 2, with central separating curve δ that separates S into two punctured tori L and R as in figure 2. Let $\varphi \in \mathrm{Mod}(S)$ be a partially pseudo-Anosov element, preserving the isotopy class of δ, isotopic to the identity on L, and pseudo-Anosov on R. An example is given by the product $\tau_1 \circ \tau_2$ of Dehn twists about α_2 and β_2, as in example 2 of section 1. We consider iteration of φ on Bers slice B_Y.

Theorem 5.1 (The half-pseudo-Anosov example). *Let $\varphi = \tau_1 \circ \tau_2$, and let $Q(\varphi^n X, Y) = Q_n$ range in the Bers slice B_Y. Then*

1. *any algebraically convergent subsequence converges to a limit Q_Y that is partially degenerate, and*
2. *any geometric limit Γ_G of the images $\{\rho_n(\pi_1(S))\}$ of the corresponding convergent representations has Kleinian manifold*

$$\overline{M_G} \cong S \times I - (R, 1/2).$$

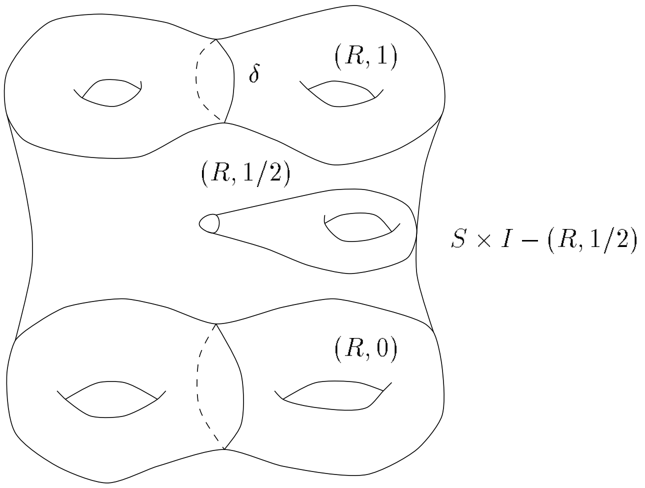

Figure 7. The topological type of $\overline{M_G}$.

Sketch of proof: We describe the steps in the proof of this special case. Again, theorem 1.1 applies to show algebraic and geometric convergence, but we pass to subsequences for logical consistency.

I. Iterate φ on Bers' slice: let μ^u be the unstable lamination of the pseudo-Anosov element $\varphi|_R \in \mathrm{Mod}(R)$. Mimicking the general pseudo-Anosov case, we have

$$\mathrm{length}_{Q_n}(\mu^u) \to 0$$

as $n \to \infty$. Let $Q_Y = \mathbb{H}^3/\rho_\infty(\pi_1(S))$ be any accumulation point of the sequence. By theorem 4.3, the support $\mathrm{supp}(\mu^u)$ is non-realizable in the limit Q_Y. This implies that either Q_Y is totally degenerate or Q_Y has an accidental parabolic.

Moreover, if there is an accidental parabolic γ, then $i(\gamma, \delta) = 0$: otherwise either $i(\gamma, \alpha_2)$ or $i(\gamma, \beta_2)$ would be non-zero, and no power of φ preserves any such isotopy class.

II. The group $\rho_\infty(\pi_1(L))$ is quasi-Fuchsian. Since any two isotopy classes (γ, η) of simple closed curves in L are stabilized by φ, lemma 3.2 applies, and some subsurface $L' \subset S$ containing L determines a quasi-Fuchsian group $\rho_\infty(\pi_1(L'))$. Since Q_Y cannot itself be quasi-Fuchsian, $\partial L'$ is non-empty. Since, however, $\partial L'$ must consist of accidental parabolics, any curve $\gamma \subset \partial L'$ must have zero intersection number with δ, α_2 and β_2. The only such curve is δ itself. Thus δ is the unique accidental parabolic of Q_Y.

III. The group $\rho_\infty(\pi_1(R))$ is totally degenerate. Having established that δ is an accidental parabolic, it follows that $Q_R = \mathbb{H}^3/\rho_\infty(\pi_1(R))$ is itself an element of $AH(R)$. Since the lamination μ^u is still non-realizable in this cover, Q_R cannot be quasi-Fuchsian. The quasi-isometric realization of φ on Q_Y lifts to a quasi-isometric realization of $\varphi|_R$ on Q_R, no power of which preserves any non-peripheral essential isotopy class. Thus Q_R has no accidental parabolics, and it follows that Q_R is totally degenerate.

IV. The manifold Q_Y is partially degenerate. Since $\rho_\infty(\pi_1(L))$ is quasi-Fuchsian and $\rho_\infty(\pi_1(R))$ is totally degenerate, the total area of the conformal boundary ∂Q_Y satisfies

$$2\pi\chi(S) < \mathrm{area}(\partial Q_Y) < 4\pi\chi(S),$$

so Q_Y is partially degenerate.

To unearth the structure of the geometric limit Γ_G of $\{\rho_n(\pi_1(S))\}$ and thence the homeomorphism type of M_G we use a similar approach to that of section 3.

V. Iterate φ^{-1} on $AH(S)$. As before, consider

$$\varphi^{-n}(Q(\varphi^n X, Y)) = Q(X, \varphi^{-n} Y).$$

The mapping class φ determines an element $\varphi_* \in \mathrm{Aut}(\pi_1(S))$ via a choice of basepoint x, which we again choose to lie in L. The elements $\varphi^{-n}(Q(\varphi^n X, Y))$ determine representations $\rho'_n = \rho_n \circ \varphi_*^n$ whose conjugacy classes again range in a compact slice of $AH(S)$ obtained by fixing X in the first factor. Passing to yet a further subsequence we may choose conjugating elements ϕ_n such that

$$\phi_n \cdot \rho'_n \cdot \phi_n^{-1}$$

converges algebraically to a limit ρ'_∞, with quotient $Q_X = \mathbb{H}^3/\rho'_\infty(\pi_1(S))$ with the following properties:

1. $\rho'_\infty(\delta)$ is parabolic,
2. $\rho'_\infty(\pi_1(R))$ is totally degenerate, and
3. $\rho'_\infty(\pi_1(L))$ is quasi-Fuchsian.

By our choice of basepoint, φ_* acts trivially on $\pi_1(L)$ (with inclusion $\pi_1(L) \hookrightarrow \pi_1(S)$ determined by the basepoint $x \in L$) so

$$\rho_\infty|_{\pi_1(L)} = \rho'_\infty|_{\pi_1(L)},$$

from which it follows that each ϕ_n can be taken to be the identity. The remarking has not affected the image group: $\{\rho_n(\pi_1(S))\} = \{\rho'_n(\pi_1(S))\}$ as subsets of $\mathrm{PSL}_2(\mathbb{C})$. Their geometric limits (after passing to a further subsequence) must therefore be equal, and each limit $\rho_\infty(\pi_1(S))$ and $\rho'_\infty(\pi_1(S))$ sits as a subgroup of the geometric limit Γ_G. By proposition 2.7, the subgroup $\rho_\infty(\pi_1(S))$ stabilizes a component Ω_Y of Ω_G the domain of discontinuity for Γ_G with quotient $\Omega_Y/\rho_\infty(\pi_1(S)) = Y$ and likewise there is a component $\Omega_X \subset \Omega_G$ such that $\Omega_X/\rho'_\infty(\pi_1(S)) = X$. Labeling $\rho_\infty(\pi_1(S)) = \Gamma_Y$ and $\rho'_\infty(\pi_1(S)) = \Gamma_X$, we have

$$\langle \Gamma_Y, \Gamma_X \rangle < \Gamma_G.$$

VI. Klein-Maskit combination. To understand the structure of $\langle \Gamma_Y, \Gamma_X \rangle$ we again use Klein-Maskit combination (theorems 2.4 and 2.5) as in section 3 to deduce that

1. the group $\langle \Gamma_X, \Gamma_Y \rangle$ is Kleinian, and satisfies
$$\langle \Gamma_Y, \Gamma_X \rangle \cong \Gamma_Y *_{\Gamma_L} \Gamma_X,$$
2. the Kleinian manifold $\overline{M_{\langle \Gamma_Y, \Gamma_X \rangle}}$ satisfies
$$\overline{M_{\langle \Gamma_X, \Gamma_Y \rangle}} \cong S \times I - (R, 1/2),$$
3. and $\partial M_{\langle \Gamma_X, \Gamma_Y \rangle} = X \sqcup Y$.

VII. $\langle \Gamma_Y, \Gamma_X \rangle = \Gamma_G$. As before, proposition 2.7 guarantees that the natural covering

$$M_{\langle \Gamma_Y, \Gamma_X \rangle} \to M_G$$

extends to an embedding on each Riemann surface X and Y in $\partial M_{\langle \Gamma_Y, \Gamma_X \rangle}$. It follows that the covering is a homeomorphism, and thus

$$\langle \Gamma_X, \Gamma_Y \rangle = \Gamma_G.$$

This completes the construction. □

The geometric limit quotient $M_G = \mathbb{H}^3/\Gamma_G$ has convex core of infinite volume: each algebraic limit $Q_X = \mathbb{H}^3/\Gamma_X$ and $Q_Y = \mathbb{H}^3/\Gamma_Y$ has developed a new simply degenerate end E_R and E'_R of its pared submanifold, cut off by surfaces isotopic into $R \subset S$. The above shows that neighborhood of these infinite volume ends embeds in under the covering projection $Q_Y \to M_G$. In effect, the surface R has "receded to infinity" in the geometric limit, much in the same way that the curve δ receded to infinity in the case of iteration of a Dehn twist. A caricature for the convex core core(M_G) that is somewhat more suggestive of this phenomenon than figure 7 appears in figure 8.

Iteration of general elements $\varphi \in \mathrm{Mod}(S)$ on a Bers slice can be understood in terms of these examples. The remaining difficulty in the proof of theorem 1.1 lies in determining the quasi-isometry type of the new geometrically tame ends arising in the algebraic and geometric limit. For details, see [**Br**], [**Mc2**], [**CT**].

6. Computer generated examples

In [**Bers2**] Bers posed the problem of producing an explicit example of a totally degenerate group on a Bers boundary. In [**Mc2**] C. McMullen obtains a numerical solution by an iterative computer algorithm: for T a punctured torus, a pseudo-Anosov mapping class $\psi \in \mathrm{Mod}(T) = \mathrm{SL}_2(\mathbb{Z})$ is specified, and the algorithm

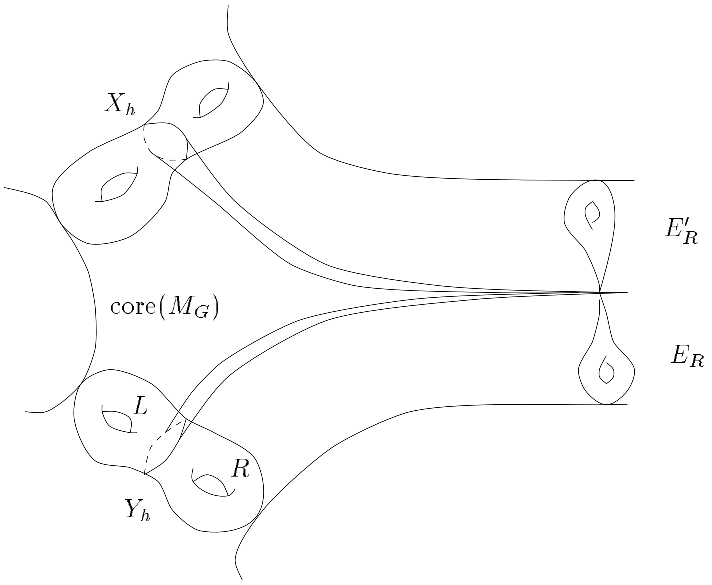

Figure 8. The hyperbolic 3-manifold M_G.

produces a totally degenerate group $\Gamma_\infty \in \partial B_R$ that is the limit of the iteration $Q(\psi^n R, \overline{R})$ for some $R \in \mathrm{Teich}(T)$ (see also [**Mc3**]).

These groups are two-generator Kleinian groups: $\Gamma_\infty = \langle \alpha, \beta \rangle$, and the algorithm chooses the unique limit of iteration of ψ such that $\mathrm{tr}(\alpha) = 3$ [**Mc3**]. In this section we apply this algorithm together with explicit Klein-Maskit combination to construct the algebraic and geometric limits of iteration of a half-pseudo-Anosov mapping class φ. This is useful for obtaining computer illustrations of their limit sets, which appear in figures 9 and 10.

The algebraic limit. Let Γ_R be so obtained, by iteration of the pseudo-Anosov element

$$\psi = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$$

on the Bers slice B_R.[4] The group Γ_R is the image of a discrete faithful representation $\rho_R \colon \pi_1(T) \to \mathrm{PSL}_2(\mathbb{C})$. Letting $\pi_1(T) = \langle a, b \rangle$, the image of the commutator $\rho_R([a, b]) = \gamma$ is parabolic. We may conjugate Γ_R in $\mathrm{PSL}_2(\mathbb{C})$ so that

$$\gamma(z) = \frac{z}{z+1},$$

and so that the limit set Λ_R of Γ_R lies entirely in $\overline{\mathbb{H}}$, and so that $(\mathbb{R} - \{0\}) \cup \infty/\gamma$ embeds in the covering projection from Ω_R/γ to Ω_R/Γ_R, where Ω_R is the domain of discontinuity of Γ_R.

[4] In this section we take $\varphi = \tau_1^4 \circ \tau_2$ rather than $\tau_1 \circ \tau_2$ to generate a slightly more interesting picture.

Our strategy for constructing the algebraic limit of section 5 will be to *combine* Γ_R with a Fuchsian punctured torus

$$\Gamma(L, \overline{L}) \in B_L,$$

$L \in \mathrm{Teich}(T)$ along a rank-1 parabolic subgroup (as in [**Mask**]).

To this end, let Γ_L be the square Fuchsian punctured torus group acting on the unit disk, i.e. with fundamental domain the ideal quadrilateral with vertices at $(1, i, -1, -i)$. Let

$$\rho_L : \pi_1(T) \to \mathrm{PSL}_2(\mathbb{C})$$

be such that $\rho_L(\pi_1(T)) = \Gamma_L$ and $\rho_L(a) = \alpha$ and $\rho_L(b) = \beta$ are automorphisms of the disk identifying pairs of sides of this quadrilateral. We pass to a conjugate representation in $\mathrm{PSL}_2(\mathbb{C})$ so that

1. Γ_L acts with limit set Λ_L the unit circle centered at $-i$,
2. $\rho_L([a, b]) = \gamma$ in $\mathrm{PSL}_2(\mathbb{C})$,
3. the curve $(\mathbb{R} - \{0\}) \cup \infty/\gamma$ embeds in the covering projection from $\Omega_L/\gamma \to \Omega_L/\Gamma_L$,
4. and so that Γ_L is symmetric with respect to an elliptic rotation by π fixing 0 and $-2i$.

The groups Γ_L and Γ_R share $\langle\gamma\rangle$ as a common rank-1 parabolic subgroup $\langle\gamma\rangle$. The compactified real line $\mathbb{R} \cup \infty$, which we will call the "equator," is an invariant round circle for the action of γ that separates the limit sets $\Lambda_L \cap \mathbb{L}$ (where $\mathbb{L}$ is the lower-half-plane) and $\Lambda_R \cap \mathbb{H}$. Since the curve $(\mathbb{R} - \{0\}) \cup \infty/\gamma$ embeds in each quotient, the combination theorem [**Mask**, p. 622] applies and we conclude

$$\langle \Gamma_R, \Gamma_L \rangle \cong \Gamma_R *_{\langle\gamma\rangle} \Gamma_L$$

which is isomorphic to the surface group $\pi_1(S)$ where S is a surface of genus 2. Label $\Gamma_Y = \langle \Gamma_R, \Gamma_L \rangle$. The limit set Λ_Y of Γ_Y appears in figure 9 after a change of coordinates sending the disk of radius 1 centered at $-i$ to the complement of the disk of radius 2 at the origin and sending 0 to $-2i$.

The above combination may be viewed as a combination of the representations ρ_R and ρ_L to form a representation

$$\rho_Y : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$$

that is faithful with discrete image Γ_Y, and thence defines an element of $AH(S)$. The group Γ_Y has an invariant component Ω in its domain of discontinuity Ω_Y, and we let $Y = \Omega/\Gamma$. In [**Br**] we show that for any X, ρ_Y is quasi-conformally conjugate to the algebraic limit of ρ_∞ of the iteration

$$\{Q(\varphi^n X, Y)\}$$

by a conjugacy that is conformal on $\Omega = \widetilde{Y}$. Here we have taken $\varphi|_L \simeq id$ and $\varphi|_R \simeq \psi$ as before.

The geometric limit. We reason using the geometric limit to conclude that there is some X such that

$$\lim_{n \to \infty} Q(\varphi^n X, Y) = \mathbb{H}^3 / \rho_Y(\pi_1(S))$$

in $AH(S)$.

We may invert the above construction to obtain a group Γ_X as the combination of the same Fuchsian punctured torus Γ_L with the limit Γ'_R of the iteration

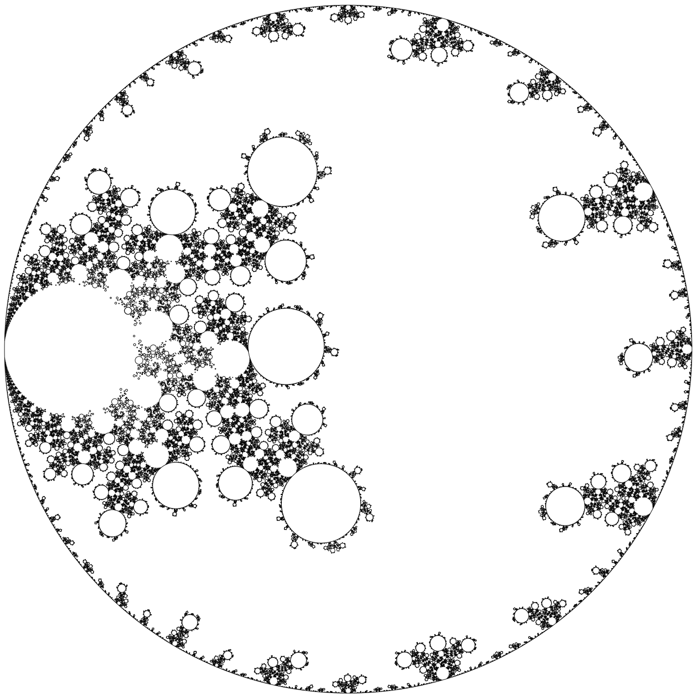

Figure 9. The limit set of the algebraic limit of half-pseudo-Anosov iteration.

$Q(R', \psi^{-n}\overline{R'})$ of ψ^{-1} on the second factor ($R' \in \text{Teich}(T)$), along a common rank-1 parabolic subgroup.

Once again, Γ_X is the image of a representation

$$\rho_X : \pi_1(S) \to \text{PSL}_2(\mathbb{C})$$

in $AH(S)$, and there is an invariant component $\Omega' \subset \Omega_X$ whose quotient Ω'/Γ_X we label by X. The group Γ_X is quasi-conformally conjugate to the limit ρ'_∞ of the iteration

$$Q(X, \varphi^{-n}Y)$$

by a conjugacy that is conformal on $\Omega' = \widetilde{X}$.

By construction, the limit set Λ_X of Γ_X lies entirely outside the open disk of radius 1 centered at $-i$, D_{-i} as does Λ_Y. Now conjugate Γ_X by an elliptic transformation with fixed points 0 and $-2i$ that rotates by π. This conjugation induces an automorphism of Γ_L (by symmetry), and the new group, which we relabel Γ_X, has limit set lying entirely inside the closed disk D_{-i}. Evidently the groups Γ_Y and Γ_X share a common Fuchsian subgroup Γ_L and the circular limit set Λ_L separates $\Lambda_Y \cap (\widehat{\mathbb{C}} - \overline{D_{-i}})$ and $\Lambda_X \cap \text{int}\, D_{-i}$. The combination theorem 2.4 applies and the resulting Kleinian group $\Gamma = \langle \Gamma_Y, \Gamma_X \rangle$ satisfies

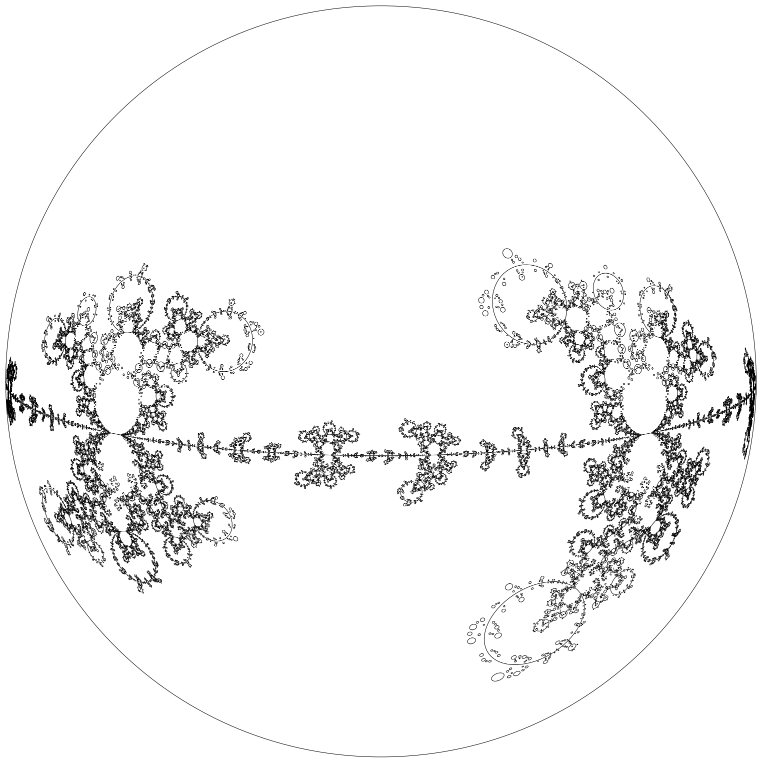

Figure 10. The limit set of the geometric limit Γ_G.

$$\Gamma \cong \Gamma_Y *_{\Gamma_L} \Gamma_X .$$

The limit set of Γ appears in figure 10 normalized so that Λ_L lies along the equator.

In [**Br**] we show that the group Γ so constructed is quasi-conformally conjugate to the geometric limit Γ_G of the iteration $Q(\varphi^n X, Y)$ (which is the same as the geometric limit of $Q(X, \varphi^{-n} Y)$). Let ϕ be such a quasi-conformal conjugacy. By the combination theorem the conformal boundary

$$\partial M_\Gamma = X \sqcup Y.$$

It follows from a theorem of Sullivan [**Sul**, Thm. V] that ϕ may be taken to be conformal, so Γ and Γ_G are conjugate in $\mathrm{PSL}_2(\mathbb{C})$.

Since the algebraic limit Γ_∞ of the iteration $Q(\varphi^n X, Y)$ is obtained up to conjugacy by passing to the stabilizer in Γ_G of the a component covering Y, it follows that $\Gamma_Y = \Gamma_\infty$ up to conjugacy.

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