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Taming the field of hyperbolic 3-manifolds

by Jeffrey F. Brock

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A hallmark of the modern study of 3-dimensional manifolds has been the role of geometry, or more specifically, *geometric structures*, in exploring their topology. W. Thurston's geometrization conjecture that each closed 3-manifold admits a decomposition into pieces each with a geometric structure is a powerful example of the force of geometry to render topological questions tractable, as its recent solution due to G. Perelman illustrates.

But last year's Clay Prize focuses on an earlier example of such a role for geometric structures on 3-manifolds, an example that beautifully draws together topology, geometry, and dynamics in low dimensions. The solution to Albert Marden's tameness conjecture represents a remarkable story of how a natural initial question can evolve and broaden in its implications and depth, cross-pollinating different subfields and disciplines along the way.

An open 3-dimensional manifold M is called *tame* if it is homeomorphic to the interior of a compact 3-manifold. In his efforts to prove Poincaré's conjecture that each simply connected closed 3-manifold M is homeomorphic to the 3-dimensional sphere, J.H.C. Whitehead discovered the first non-tame 3-manifold: a contractible 3-manifold topologically distinct from the open ball. This example led the way to numerous constructions of non-tame 3-manifolds with non-trivial fundamental group. Albert Marden, in his seminal investigation of the topological properties of 3-manifolds with a complete metric of constant negative curvature -1, the so-called *hyperbolic 3-manifolds*, formulated the following conjecture [Mar].

Marden's Tameness Conjecture — *Each complete hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold.*

The conjecture, evidently easy to state, historically well motivated, and resilient, is made all the more remarkable by its profound importance and applications to disparate branches of mathematics. Last year's Clay Prize was awarded to Ian Agol, Danny Calegari, and David Gabai for two independent solutions of Marden's conjecture [Ag], [CG].

Its implications for

1. the Ahlfors measure conjecture for limit sets of finitely generated Kleinian groups, and
2. the classification of finitely generated Kleinian groups up to conjugacy,

lend it an unusual status at the intersection of topology, geometry, and dynamical systems

The network of implications and connections placing this conjecture in its modern framework owes a great debt to the work of W. Thurston, F. Bonahon, and R. Canary. We will attempt to elucidate this web of dependencies, describe the ramifications of the conjecture, now established, and to give a perspective on its proof.

A few words about Kleinian groups. At the center of the discussion is the notion of a Kleinian group Γ, namely, a discrete subroup of $\mathrm{PSL}_2(\mathbb{C})$, which plays alternatively the role of a group of Möbius transformations of $\widehat{\mathbb{C}}$ and a group of orientation-preserving isometries of hyperbolic 3-space $\mathbb{H}^3$ via the natural extension to the unit-ball model of $\mathbb{H}^3$. As the present discussion concerns groups with a manifold quotient $M = \mathbb{H}^3/\Gamma$, we will assume Γ is torsion free, and we will for simplicity omit any discussion of the case when Γ has parabolic elements, which correspond to embedded *cusp-regions* of M with standard topology.

The action of Γ on the Riemann sphere $\widehat{\mathbb{C}}$ determines a partition $\widehat{\mathbb{C}} = \Lambda \sqcup \Omega$ of the sphere into its *limit set* $\Lambda = \overline{\Gamma(0)} \cap \widehat{\mathbb{C}}$, the smallest closed Γ-invariant subset of $\widehat{\mathbb{C}}$ and its *domain of discontinuity* Ω, where Γ acts properly discontinuously by conformal homeomorphisms.

The quotient $(\mathbb{H}^3 \cup \Omega)/\Gamma$ gives a partial boundary for the complete hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, by adjoining Ω/Γ, the *conformal boundary* of M, a finite collection of finite-type Riemann surfaces by Ahlfors' *finiteness theorem* [Ah1].

The convex hull $CH(\Lambda)$ in hyperbolic space of the limit set Λ is the smallest hyperbolically convex set in $\mathbb{H}^3$ that contains Λ in its closure. As $CH(\Lambda)$ is Γ-invariant, its quotient $CH(\Lambda)/\Gamma$ is a geometrically preferred subset $C(M)$ of M, the *convex core* of M.

Ahlfors' conjecture. One of the more remarkable features of Marden's conjecture is its implication for conformal dynamical systems, established by Thurston, Bonahon, and Canary some years after its formulation.

AHLFORS' MEASURE-ZERO CONJECTURE — *If the limit set Λ of a finitely generated Kleinian group Γ is not all of $\widehat{\mathbb{C}}$, Λ has zero Lebesgue measure in $\widehat{\mathbb{C}}$.*

The conjecture was later expanded to include the expectation of ergodicity of the action of Γ on Λ if $\Lambda = \widehat{\mathbb{C}}$. Ahlfors' motivation rested on questions in the quasi-conformal deformation theory of Kleinian groups. For if each such limit set has measure zero, a non-trivial quasi-conformal deformation of a Kleinian group induces a non-trivial quasi-conformal deformation of its conformal boundary, guaranteeing that this deformation theory rests on a proper understanding of the Teichmüller theory of the conformal boundary.

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But it is perhaps lucky that Ahlfors did not have access to modern computer renderings of the limit sets of finitely generated Kleinian groups, as the conjecture might never have been made in the first place (see Figure 1).

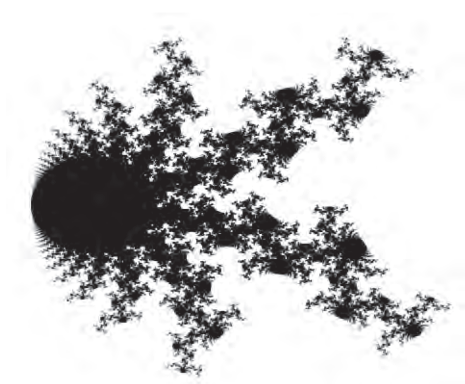

Figure 1.
A measure-zero limit set of a degenerate group.

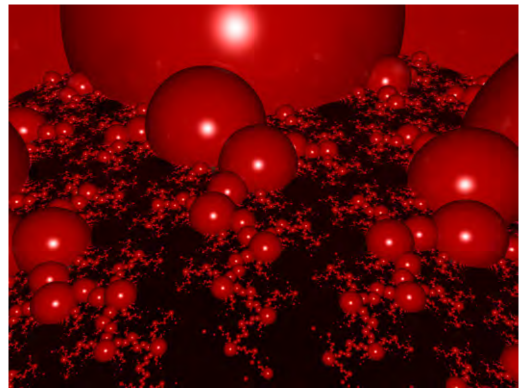

Figure 2. Inside the convex hull of the limit set.

Marden's conjecture. Marden set out to give a coherent description of the topological type of the quotient spaces $\mathbb{H}^3/\Gamma$ of finitely generated Kleinian groups. In the *geometrically finite* setting, when the convex core is assumed to have a finite volume unit neighborhood, his proof that the quotient manifold is homeomorphic to the interior of the compact manifold represented the first deep investigation of this topological question [Mar].

It was in this article that he formulated the tameness conjecture, predicting that such a simple topological description should exist for any finitely generated Kleinian group Γ. This conjecture would later be proven in more general cases and in turn be reinterpreted to serve as a centerpiece of Thurston's conjectural classification of finitely generated Kleinian groups. The conjecture grows out of the *core theorem* of Peter Scott [Sco] that each 3-manifold with finitely generated fundamental group admits a compact submanifold whose inclusion is a homotopy equivalence. In the setting of a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, then, finite generation of Γ guarantees the existence of a decomposition into a compact submanifold and a finite collection of complementary pieces, termed *ends* of the manifold (more properly, each piece is a *neighborhood of an end*, of M, depending on the choice of core).

To prove Marden's conjecture amounts to proving a choice of compact core $\mathscr{M}$ exists so that each end $\mathscr{E}$ is a product $S \times \mathbb{R}^+$ where S is a component of the boundary $\partial\mathscr{M}$. In the geometrically finite setting, a nearest point retraction map from the conformal boundary Ω/Γ provided a natural product structure for the ends of $\mathbb{H}^3/\Gamma$, which later came to be called *geometrically finite ends*.

Wild ends and tame ends. It is instructive to review what can go wrong in general. We briefly review Whitehead's construction: consider S^3 decomposed into solid tori U and V along a common torus boundary component T. Let $h\colon V \to V$ denote the pictured embedding of V

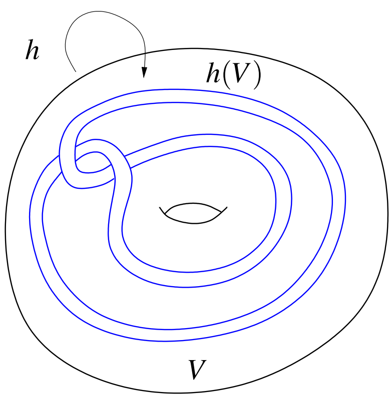

Figure 3. Whitehead's embedding.

into itself: $\mathrm{int}(V)\setminus h(V)$ is then the complement of the Whitehead link in S^3. A meridian J of V is then homotopically essential in $V\setminus h(V)$ but bounds a disk in $S^3\setminus h(V) = U\cup(V\setminus h(V))$.

Then $J, h(J), h^2(J), \ldots, h^n(J)$ denote loops that are homotopically distinct and non-trivial in $V\setminus h^n(V)$, all of which become trivial in $S^3\setminus h^n(V)$.

Letting $X_\infty = \cap_n h^n(V)$, the *Whitehead manifold* $S^3\setminus X_\infty$ is a simply connected, indeed contractible 3-manifold with the property that the removal of a compact submanifold V leaves a 3-manifold with infinitely generated fundamental group (by an application of Van Kampen's theorem). Such a manifold cannot be homeomorphic to $\mathrm{int}(B^3)$.

Note, however, that by the Cartan-Hadamard theorem, there is a unique simply connected complete Riemannian manifold of constant curvature -1, namely, hyperbolic 3-space $\mathbb{H}^3$. So the assumption of negative curvature tames the topology in this case.

What about lifting the assumption of simple connectivity? In a variant of Whitehead's construction due to Tucker [Tck], one replaces U and V with a handlebodies of the same genus, and $h\colon V\to V$ becomes a knotted embedding of V into itself (h is homotopic but not isotopic to the identity). The result is an open 3-manifold that is *exhausted by compact cores*, but for which the complement of U has infinitely generated fundamental group. The possibility

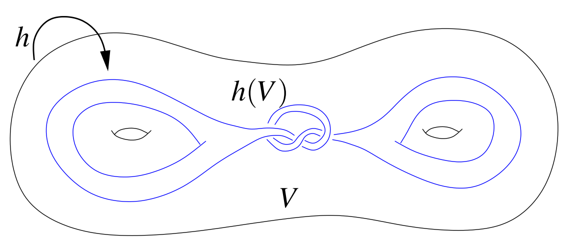

Figure 4. A knotted handlebody.

that such a 3-manifold might admit a complete hyperbolic structure was later ruled out by a result of Souto [Sou], but examples of sequences of embeddings with unbounded genus and more

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and more complicated knotting remained out of reach until the solution of the full conjecture.

Geometric tameness. Though Ahlfors established his measure-zero conjecture for geometrically finite Kleinian groups [Ah2], W. Thurston described an extension of Ahlfors' theorem in his Princeton Lecture Notes *Geometry and Topology of Three-Manifolds* [Th1], employing very directly the internal geometry of the quotient hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ where Γ is a finitely generated torsion-free Kleinian group.

A centerpiece of Ahlfors' argument involves the natural harmonic extension of the characteristic function of a measurable set on the Riemann sphere $\widehat{\mathbb{C}}$ to a harmonic function $\tilde{h}$ on hyperbolic space $\mathbb{H}^3$, the solution to the *Dirichlet problem*. If the set is invariant by the action of the Kleinian group Γ, its characteristic function and hence its extension are as well, and $\tilde{h}$ descends in the quotient to a harmonic function h on the hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$. Harmonicity of h guarantees that its gradient flow is volume preserving.

A geometric condition introduced by Thurston that generalizes geometric finiteness, termed *geometric tameness*, ensures that no positive measure set of flow lines can exit an end of the convex core $C(M)$. This gives a maximum principle for non-constant harmonic functions on $C(M)$, showing that no measurable invariant subset of Λ can have a point of Lebesgue density unless it is all of $\widehat{\mathbb{C}}$.

Briefly, the assumption of geometric tameness for the end $\mathscr{E}$ of $C(M)$ provides for intrinsically hyperbolic surfaces X_n exiting the end $\mathscr{E}$, each homotopic to the boundary S of $\mathscr{E}$; the geometry of X_n guarantees a bounded-diameter statement that controls the volume swept out by a unit neighborhood of each X_n, which in turn serves to limit the measure of flow lines of ∇h crossing X_n. The manifold $\mathbb{H}^3/\Gamma$ is geometrically tame if all its ends are geometrically finite (asymptotic to a component of the conformal boundary) or geometrically tame.

Topological and geometric tameness. Thurston established the topological implications of geometric tameness in many settings in his Notes [Th1]. He used the surfaces exiting the end to build a topological product structure for the end, proving *topological tameness* in these settings. A key role is played by sequences of closed geodesics $\{\gamma_n\}$ that exit $\mathscr{E}$ that can be made *simple* (non self-intersecting) when realized as curves on the boundary S. Such geodesics serve to anchor the surfaces X_n and force them to infinity.

Thurston's geometric condition, moreover, gave rise to a new invariant of the geometry of a geometrically tame end, namely, its *ending lamination*, a kind of limit of $\{\gamma_n\}$. A posteriori, the ending lamination $\nu(E)$ encodes the limiting combinatorial picture of bounded length closed geodesics that exit the end E. It is only clearly well defined, however, when the end is assumed geometrically tame.

In ensuing years, Bonahon showed that one can always find closed geodesics $\{\gamma_n\}$ that may fail the simplicity assumption above, and showed how to use their limit to find new geodesics $\{\gamma_n'\}$ that do satisfy the simplicity condition, provided that $S \hookrightarrow M$ is injective on the level of fundamental group [Bon]. Geometric tameness followed in this case. Later, Canary showed how to relax Bonahon's hypotheses in such a way to show that any topologically tame hyperbolic 3-manifold is *geometrically tame*, thereby reducing Ahlfors' original conjecture to Marden's [Can1]. He also showed how to generalize Thurston's argument to build topological product structures for general geometrically tame ends, thereby showing the equivalence of topological and geometric tameness.

The proof. A complete description of the proof is naturally out of the scope an article such as this one. Essentially, the goal rested in trying to find the appropriate replacement for the simple closed geodesics $\{\gamma_n\}$ exiting an end $\mathscr{E}$ that is not geometrically finite as a method to produce exiting surfaces X_n in the same homotopy class. It is interesting to note both how geometry and topology played key roles:

1. The original argument of Bonahon guarantees the existence of closed geodesics $\{\gamma_n\}$ exiting the end $\mathscr{E}$, not necessarily homotopic to simple curves on S.
2. A topological innovation called an *end-reduction* allows one to find surfaces in the end that lie outside or "engulf" an arbitrary finite subcollection of these geodesics.
3. Such surfaces can be pulled tight to geometric surfaces in the complement of the geodesics they enclose: the sets of geodesics are said to be "shrinkwrapped" by a geometric ($CAT(-1)$) surface, and infinitely many such lie in the same homotopy class.
4. A bounded diameter lemma together with a homology argument guarantees these surfaces exit the end, and then standard techniques produce the desired product structure as in the geometrically tame setting.

A critical topological insight was to notice the efficacy of technology on analyzing the ends of open 3-manifolds due to M. Brin and T. Thickstun, and R. Myers. Indeed, their *end reductions* provide the key facts from 3-manifold topology to guarantee the usefulness of the exiting geodesics — one can imagine that they provide cellophane that is used in the shrinkwrapping. In effect, the shrinkwapped surfaces Z_n replace the surfaces assumed in geometric tameness, and the exiting geodesics γ_n serve to show the surfaces Z_n exit the end $\mathscr{E}$.

We remark that we have focused on the solution presented by Calegari and Gabai, in which shrinkwrapping is accomplished using minimal surfaces, but a more recent treatment due to T. Soma employs standard polyhedral techniques to obtain the same result [So].

The Classification theorem. Much of Ahlfors' original motivation for the measure-zero conjecture was obviated in practice by Sullivan's rigidity theorem [Sul], which guaranteed the absence of deformations supported on the limit set alluded to previously. In more recent years, it

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was the progress toward and ultimate solution to the *ending lamination conjecture* of Thurston due to Minsky [Min] and concluded by work of this author with Canary and Minsky [BCM2, BCM1] that brought renewed attention to the tameness conjecture.

The ending lamination conjecture predicts that each geometrically tame hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is determined up to isometry by its homeomorphism type, its cusps (regions corresponding to *parabolic* elements of Γ), and its *end invariant* $\nu(M)$ consisting of the conformal structures on Ω/Γ and the ending laminations $\{\nu(\mathscr{E})\}$ for each geometrically tame end of $C(M)$. But in the intervening years, the aforementioned work of Bonahon and Canary showed the *topological* tameness of M to be the complementary conjecture for a complete classification theorem.

THE CLASSIFICATION THEOREM — *Each complete hyperbolic 3-manifold with finitely generated fundamental group is determined up to isometry by its topology, its cusps, and its end invariant.*

The theorem, which formally combines the tameness theorem and the ending lamination theorem, sets to rest what has been perhaps the central motivating conjectural question in finitely generated Kleinian groups. It is notable however, that the output is richer than simply a classification: the method of proof of the ending lamination theorem produces a combinatorial model for the ends M directly from the end-invariant data $\nu(M)$, and thus a uniform picture of its hyperbolic metric up to bi-Lipschitz equivalence. That any finitely generated Kleinian group can now be understood so concretely provides new methods to study the internal geometry of the full spectrum of hyperbolic 3-manifolds, their deformation spaces, and how their topological, analytic, and geometric invariants interrelate.

Implications. In his seminal Bulletin article [Th2], Thurston's list of twenty-four problems and questions set the stage for the next thirty years of activity in the geometry and topology of 3-manifolds, and in particular the fields of Kleinian groups and deformation theory of hyperbolic 3-manifolds. (It is notable that Thurston's celebrated *geometrization conjecture* is question 1 on this list. To reflect on how far the field of geometric structures on 3-manifolds has progressed in the last ten years is, once again, beyond the scope of this article, but we simply state, for emphasis, questions in Thurston's list in which the solution to tameness plays a central role.

1. AHLFORS' MEASURE CONJECTURE. Thurston's harmonic flow argument together with Canary's theorem that topologically tame implies geometrically tame guarantee that the limit set Λ of a finitely generated Kleinian group has zero or full measure, and if full, the action of Γ is ergodic on Λ.

2. THE CLASSIFICATION OF FINITELY GENERATED KLEINIAN GROUPS. Because ending laminations are only defined for tame ends, the proof of the ending lamination conjecture [BCM2, BCM1] requires

tameness to apply to all finitely generated Kleinian groups.

3. The density conjecture. After work of Namazi and Souto [NS], each candidate end-invariant can be realized in a limit of geometrically finite manifolds, and hence the limit is isometric to a given (necessarily tame) manifold by the ending lamination conjecture. This resolves the *Density Conjecture* of Bers, Sullivan, and Thurston (independently resolved by [BS]).

4. The Model Manifold Conjecture. A combinatorial bi-Lipschitz model for the ends of hyperbolic 3-manifolds with finitely generated fundamental group, conjectured by Thurston, arises directly from the ending lamination in the proof of the ending lamination conjecture, and is thus only operative in full generality after tameness.

There are numerous other deep implications of tameness and the model manifold theorem for the geometry, topology, and dynamics of finitely generated Kleinian groups and their associated hyperbolic 3-manifolds. The ergodicity of the geodesic flow on the unit tangent bundle, Simon's tameness conjecture for covers of compact manifolds, the recently claimed local-connectivity theorem for limit sets of finitely generated Kleinian groups [Mj], and the enumeration of components of the deformation space of a Kleinian group [BCM2], are just a few other major examples. For a survey of these and many other applications see [Can2].

The story of the symbiosis between geometry and topology in 3-dimensions continues to be written, and more and more precise connections between geometric and topological invariants for 3-manifolds emerge with ever-increasing frequency. The tameness theorem taken together with the model manifold theorem guarantees that for each finitely generated Kleinian group Γ, the hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ can be modeled in a combinatorial way on its ending laminations. Such a combinatorial structure provides many new methods and tools, and indeed new questions for investigation in the study of 3-manifolds.

An apocryphal story has it that Ahlfors submitted a one-line grant proposal late in his career containing the single sentence: "I will continue to try to understand the work of Thurston." No doubt he would be gratified to see how the cumulative efforts of so many mathematicians have culminated in such a rich narrative, intertwining the solution to his own conjecture with those of Marden, Thurston, Bers, and Sullivan, and how so many fundamental questions in the geometry, topology, and dynamics of Kleinian groups have been set to rest.

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References

[Ag] I. Agol. Tameness of hyperbolic 3-manifolds. *Preprint,*

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arXiv:mathGT/0405568 (2004).

[Ah1] L. Ahlfors. Finitely generated Kleinian groups. *Amer. J. of Math.* **86**(1964), 413–429.

[Ah2] L. Ahlfors. Fundamental polyhedrons and limit point sets of Kleinian groups. *Proc. Nat. Acad. Sci. USA* **55**(1966), 251–4.

[Bon] F. Bonahon. Bouts des variétés hyperboliques de dimension 3. *Annals of Math.* **124**(1986), 71–158.

[BCM1] J. Brock, R. Canary, and Y. Minsky. The classification of finitely generated Kleinian groups. *In preparation (2010).*

[BCM2] J. Brock, R. Canary, and Y. Minsky. The classification of Kleinian surface groups, II: the ending lamination conjecture. *Submitted (2004).*

[BS] K. Bromberg and J. Souto. Density of Kleinian groups. *In preparation*.

[CG] D. Calegari and D. Gabai. Shrinkwrapping and the taming of hyperbolic 3-manifolds. *J. AMS* **19**(2006), 385–446.

[Can1] R. D. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **6**(1993), 1–35.

[Can2] R. D. Canary. Marden's tameness conjecture: history and applications. In *Geometry, Analysis and Topology of Discrete Groups*, volume 6 of *Adv. Lect. Math. (ALM)*, pages 137–162. Int. Press, Somerville, MA. (2008).

[Mar] A. Marden. The geometry of finitely generated Kleinian groups. *Annals of Math.* **99**(1974), 383–462.

[Min] Y. Minsky. The classification of Kleinian surface groups I: A priori bounds. *Preprint,* arXiv:math.GT/0302208. *To appear, Annals of Math.* (2010).

[Mj] M. Mj. Cannon-Thurston maps for Kleinian surface groups. *Preprint,* arXiv:math/0607509 (2010).

[NS] H. Namazi and J. Souto. Non-realizability and ending laminations: the proof of the density conjecture. *In preparation.* (2010).

[Sco] P. Scott. Compact submanifolds of 3-manifolds. *J. London Math. Soc.* **(2)7**(1973), 246–250.

[So] T. Soma. Existence of ruled wrappings in hyperbolic 3-manifolds. *Geom. Top.* **10**(2006), 1173–1184.

[Sou] J. Souto. A remark on the tameness of hyperbolic 3-manifolds. *Topology* **44**(2005), 459–474.

[Sul] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. In *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference*. *Annals of Math. Studies* 97, Princeton (1981).

[Th1] W. P. Thurston. *Geometry and Topology of Three-Manifolds*. Princeton lecture notes (1979).

[Th2] W. P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. AMS* **6**(1982), 357–381.

[Tck] T. W. Tucker. Non-compact 3-manifolds and the missing-boundary problem. *Topology* **13**(1974), 267–273.