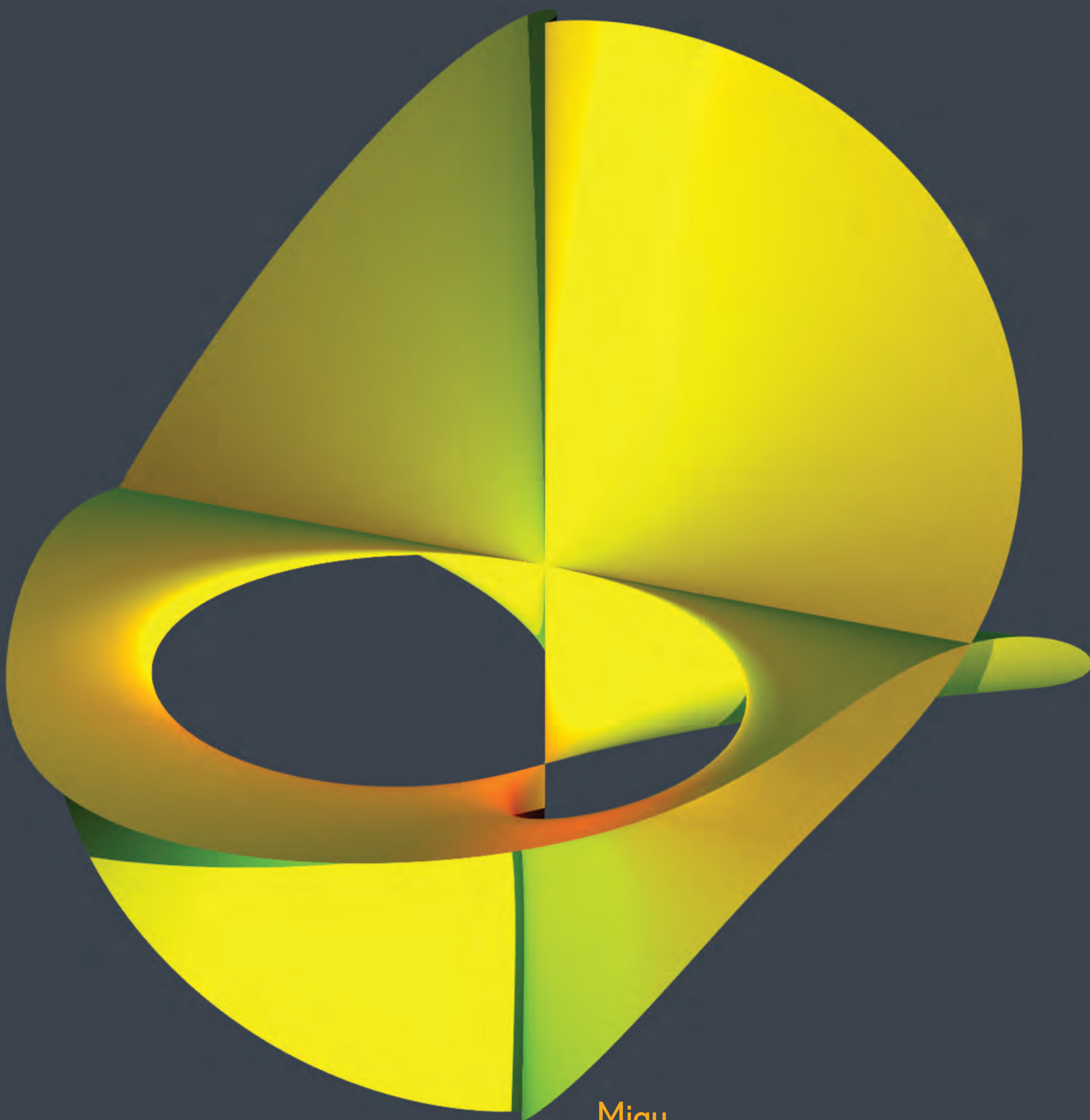
annual report

CLAY MATHEMATICS INSTITUTE



Miau

2009

Clay Research Awards
Thurston's Geometrization Conjecture
Clay Lectures on Mathematics

Nominations, Proposals, and Applications

Nominations for Senior and Research Scholars are considered four times a year at our Scientific Advisory Board (SAB) meetings. Principal funding decisions for Senior Scholars are made at the September SAB meeting. Additional nominations will be considered at other times as funds permit. Clay Research Fellow nominations are considered once a year and must be submitted according to the schedule below:

Nomination Deadlines

Senior Scholars: August 1
Research Fellows: October 30

Address all nominations to the attention of the assistant to the president at nominations@claymath.org.

Nominations may also be mailed to:

Clay Mathematics Institute
One Bow Street
Cambridge, MA 02138

The Clay Mathematics Institute invites proposals for conferences and workshops. Proposals, which need not be long, will be judged on their scientific merit, probable impact, and potential to advance mathematical knowledge.
Our budget is often committed well in advance, so please submit applications at your earliest convenience.
A budget and standard cover sheet should be sent well in advance to the attention of the assistant to the president at proposals@claymath.org.

Proposal Deadlines

Workshops & Conferences: August 1, February 15
Bow Street Workshops: 3 months prior

Proposals may also be mailed to:

Clay Mathematics Institute
One Bow Street
Cambridge, MA 02138

Please find more information and the standard cover sheet at www.claymath.org/proposals.
Noteworthy proposals will be considered at other times. However, most funding decisions will be made in accordance with the deadlines above.

CMI's One Bow Street library holdings include two collections of mathematical books that were acquired as gifts by the Institute:

Raoul Bott Library, gift received from the Bott family in 2005.

701 volumes consisting of books, journals, and preprints on topology, geometry, and theoretical physics.

George Mackey Library, gift received from the Mackey family in 2007.

1,310 volumes consisting of books and periodicals related to quantum mechanics, group representations, and physics, in addition to titles on a wide range of historical, philosophical, and scientific topics.

CMI's Digital Library includes the following facsimiles of significant historical mathematical books and manuscripts that are accessible online at www.claymath.org/library/historical:

Euclid's Elements, Constantinople, 888 AD (Greek). MS at the Bodleian Library.

The oldest extant manuscript and printed editions of Euclid's Elements, in Greek (888 AD) and Latin (1482 AD), respectively. High resolution copies of the manuscript are available for study at the Bodleian Library, Oxford University and at the Clay Mathematics Institute, Cambridge, Massachusetts. Full online editions are available at CMI, Libraries without Walls, and rarebookroom.org.

Euclid's Elements, first printed edition, 1482 AD (Latin)

The first printed edition of Euclid's Elements, Elementarum Euclidis, appeared in Venice in 1482 through the work of Aldus Manutius. See Libraries without Walls or rarebookroom.org.

Riemann's 1859 Manuscript

The manuscript in which Riemann formulated his famous conjecture about the zeroes of the zeta function.

Felix Klein Protokolle

The Klein Protokolle, comprising 8,600 pages in twenty-nine volumes, records the activity of Felix Klein's seminar in Goettingen for the years 1872-1912.

Clay Mathematics Institute

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Letter from the President

Dear Friends of Mathematics

James A. Carlson, President

The increase in mathematical knowledge in the last few years has been nothing short of phenomenal. To highlight some examples: the work of Green and Tao on arithmetic progressions in the prime numbers, the proof of the fundamental lemma of the Langlands program by Laumon and Ngô, the resolution of the Ahlfors and Marden conjectures by Agol and Calegari-Gabai, the work of Waldspurger in p-adic harmonic analysis, the extension of the theory of minimal models from dimension three to arbitrary dimension by Hacon and McKernan with Birkar and Cascini, the proof of the Sato-Tate conjecture by Harris and Taylor in collaboration with Clozel and Shepherd-Barron, and the astounding breakthrough of Perelman who, armed with the pioneering work of Richard Hamilton on Ricci flow, as well as ideas and techniques in Riemannian geometry, resolved not only the Poincaré conjecture, but also William Thurston's geometrization conjecture. Because of limitations of space, and, more importantly, the author's knowledge, one has to end a list at some point, necessarily omitting many developments that deserve mention. But the list does drive home the point of the opening sentence, and it demonstrates the remarkable vigor and dynamism of our subject.

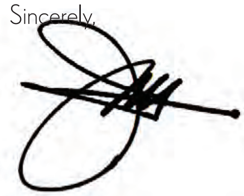
Such a list raises the question: have we reached a high point, is this work the product of a great generation, the likes of which we will not see again? There are two parts to the query. First, is the geometry of possible mathematical knowledge like that of the sphere, finite and in principle completely accessible as is the surface of our earth? Or is it like the Euclidean or hyperbolic plane, infinite in extent, with the disk representing what is known ever increasing in both area and perimeter?

Beyond the perimeter lie great landmarks such as the Riemann hypothesis. We hope, as Hilbert did, that one day we will reach each of them—the ones already visible such as Riemann's famous question, and those whose nature and location can not now be guessed, even by those among us who are most far-sighted.

The question about the nature of the frontier, more philosophical than scientific, can perhaps never be properly formulated, let alone answered. But the feeling one gets from the course of mathematical history is that the circle dividing the dark from the light is one that not only grows, but accelerates. Thus if pressed, I would bet on the Euclidean or hyperbolic geometry. The second part of the question is a human one—is there enough talent going into mathematics to sustain the intellectual enterprise? One cannot know the long-term future, but in the short term the answer is a resounding yes. The ability and command of current knowledge of the younger generation is extraordinary—so much so that I am glad that I applied to graduate school when I did, over four decades ago!

In this annual report you will find articles about some of the developments mentioned above: James Arthur's commentary on the work of Jean-Loup Waldspurger, Jeff Brock's article on the solution of the Ahlfors and Marden conjectures, and an article by David Gabai and Steve Kerckhoff on Thurston's geometrization conjecture. We do live in a golden age of mathematics!

Sincerely,



James A. Carlson
President

Clay Research Conference

The third Clay Research Conference, an event devoted to recent advances in mathematical research, was held at Harvard on May 4 and 5 at the Harvard Science Center (Hall E). The lectures, listed below, covered a wide range of fields: algebraic geometry, complex analysis, harmonic analysis, number theory, dynamical systems, and topology.

Conference speakers were Herwig Hauser (University of Vienna), Heisuke Hironaka (Seoul National University), Peter Jones (Yale University), Curtis T. McMullen (Harvard University), Yair Minsky (Yale University), Dinakar Ramakrishnan (Caltech), Kannan Soundararajan (Stanford University), and Jean-Loup Waldspurger (Institut de mathématiques de Jussieu). Abstracts of their talks are given below. Videos of the talks are available on the Clay Mathematics Institute website, at www.claymath.org/video.

On the afternoon of May 4, the Clay Research Awards were presented to Jean-Loup Waldspurger and to Ian Agol together with Danny Calegari and David Gabai. The citations read:

Jean-Loup Waldspurger
(Institut de mathématiques
de Jussieu)
For his work in p-adic
harmonic analysis.

Ian Agol (UC Berkeley),
Danny Calegari (Caltech),
and David Gabai (Princeton),
for their solutions of the
Marden Tameness Conjecture.

Previous recipients of the award, in reverse chronological order are:

2008 Cliff Taubes (Harvard University)
Claire Voisin (Institut de Mathématiques de Jussieu, CNRS, IHES)

2007 Alex Eskin (University of Chicago)
Christopher Hacon (University of Utah) and
James McKernan (UC Santa Barbara)
Michael Harris (Université de Paris VII) and
Richard Taylor (Harvard University)

2005 Manjul Bhargava (Princeton University)
Nils Dencker (Lund University, Sweden)

2004 Ben Green (Cambridge University)
Gérard Laumon (Université de Paris-Sud, Orsay) and
Bao-Châu Ngô (Université de Paris-Sud, Orsay)

2003 Richard Hamilton (Columbia University)
Terence Tao (University of California, Los Angeles)

2002 Oded Schramm (Theory Group, Microsoft Research)
Manindra Agrawal (Indian Institute of Technology, Kanpur)

2001 Edward Witten (Institute for Advanced Study)
Stanislav Smirnov (Royal Institute of Technology, Stockholm)

2000 Alain Connes (College de France, IHES, Vanderbilt University)
Laurent Lafforgue (Institut des Hautes Études Scientifiques)

1999 Andrew Wiles (Princeton University)

The Clay Mathematics Institute presents the Clay Research Award annually to recognize major breakthroughs in mathematical research. Awardees receive the bronze sculpture "Figureight Knot Complement VII/CMI" by Helaman Ferguson and are named Clay Research Scholars for a period of one year. As such they receive substantial, flexible research support. Awardees have used their research support to organize a conference or workshop, to bring in one or more collaborators, to travel to work with a collaborator, and for other endeavors.

Herwig Hauser delivering his talk at the conference.

Clay Research Conference

Herwig Hauser
University of Vienna

Resolution of singularities in zero and positive characteristic

Hauser discussed the principal ideas in the proof of resolution of singularities in characteristic zero, the main obstructions to applying these ideas to the case of positive characteristic, and recent approaches to overcome these obstructions.

Heisuke Hironaka
Harvard University and Kyoto University

Resolution of singularities in algebraic geometry

Hironaka presented his approach to proving resolution of singularities of an algebraic variety of any dimension over a field of any characteristic. Parts of this approach are as interesting from a technical and conceptual standpoint as they are from the standpoint of the end result. The resolution problem for all arithmetic varieties (meaning algebraic schemes of finite type over the ring of integers) is reduced to the question of how to extend the result from modulo pm to modulo pm+1 after a resolution of singularities over Q. Problems that arise in this approach were discussed.

Peter Jones
Yale University

Some remarks on SLE and an extended Sullivan dictionary

The Sullivan dictionary translates statements about Kleinian groups into statements about Julia sets and vice versa. For example, a limit set on the Kleinian group side corresponds to a Julia set, and the orbit of a point under a Kleinian group corresponds to the inverse images of a point by a rational map. We discuss adding another category to the dictionary, namely SLE, Schramm-Loewner Evolution. Here limit sets and Julia sets correspond to the SLE "trace." We point out that with suitable modifications, the Sullivan dictionary can be enlarged to include SLE. As an example, we discuss the various analogues of the Ahlfors conjecture for Kleinian groups. We also discuss the various versions of rigidity that appear in the dictionary. This lecture is aimed at a general mathematical audience; we stress ideas, not technicalities of the proofs.

Curtis T. McMullen
Harvard University

Billiards and moduli spaces

We discuss ergodic theory over the moduli space of compact Riemann surfaces, and its connections with algebraic geometry, Teichmüller theory, and billiard tables with optimal dynamics.

Yair Minsky
Yale University

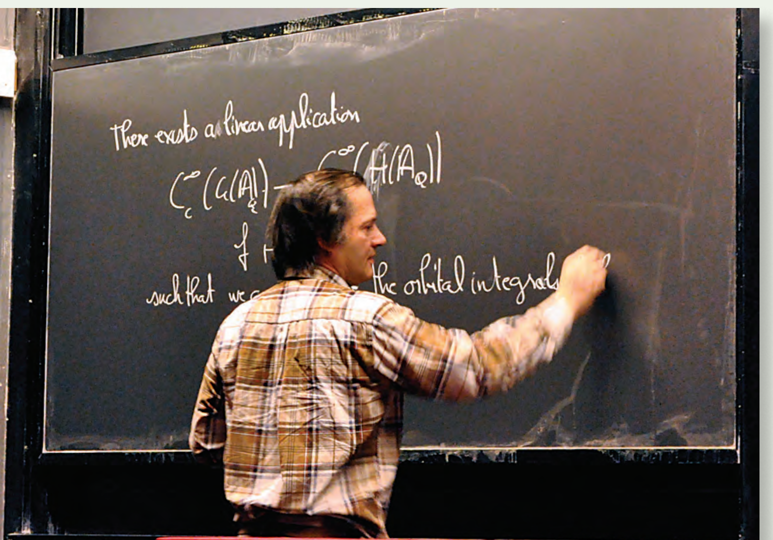
Topology and geometry of ends of hyperbolic 3-manifolds

The classification of non-compact hyperbolic 3-manifolds with finitely generated fundamental groups depends on an understanding of the topology and asymptotic geometry of their ends. A number of advances in recent years have made this classification possible, and more. I discuss the background and features of this theory, and its applications to a fuller understanding of how these manifolds (compact and non-compact) cover and approximate each other.

Dinakar Ramakrishnan
Caltech

Functoriality: ubiquity and progress

Questions in automorphic forms and number theory often get tied up with the magnificent, largely conjectural, edifice of functoriality, a simple instance being the desire to know if certain four-dimensional Galois representations occurring inside the cohomology of Siegel modular threefolds are symplectic. Of particular importance, besides base change, is the transfer of automorphic forms from orthogonal and symplectic groups to the general linear group, which sheds light on many problems. Crucial progress has been made of late in the work of Arthur via the twisted trace formula, extending the earlier results known for generic cusp forms, which had relied on the elegant converse theorem insight of Piatetski-Shapiro. Part of what makes Arthur's approach work is the incredible recent progress on the (different guises of) fundamental lemma due to Ngô, Waldspurger, and others. This talk will try to introduce the basic global statements, a few ideas, and applications.



Kannan Soundararajan

Stanford University

Quantum unique ergodicity and number theory

A fundamental problem in the area of quantum chaos is understanding the distribution of high eigenvalue eigenfunctions of the Laplacian on certain Riemannian manifolds. A particular case which is of interest to number theorists concerns hyperbolic manifolds arising as a quotient of the upper half-plane by a discrete "arithmetic" subgroup of SL2(R) (for example, SL2(Z), and in this case the corresponding eigenfunctions are called Maass cusp forms). In this case, Rudnick and Sarnak have conjectured that the high energy eigenfunctions become equi-distributed. I discuss some recent progress that has led to a resolution of this conjecture, and I discuss a holomorphic analog for classical modular forms.

Jean-Loup Waldspurger

Institut de mathématiques de Jussieu

Endoscopy and harmonic analysis on reductive groups

Let G be a connected reductive group over a number field and let H be an endoscopic group of G. A conjecture of Langlands predicts that there exists a correspondence between automorphic representations of H(A) and automorphic representations of G(A), where A is the ring of adeles of the ground field. Langlands' idea of proof is to compare the Arthur-Selberg's trace formulas of H and G. It is necessary to solve many problems, in particular two problems of harmonic analysis over a local field: the transfer conjecture and the fundamental lemma. These two questions remained open until the decisive result of Bao-Châu Ngô, achieved two years ago. In my talk, I try to explain what is the endoscopic transfer and what is the fundamental lemma. I give several statements of that lemma, more or less sophisticated. I try to explain the situation at the present time. In fact, all the useful problems are resolved even if certain related questions of harmonic analysis remain open.

Clay Research Awards

Jean-Loup Waldspurger receiving the 2009 Clay Research Award that was presented by Landon Clay and President James Carlson.

David Gabai and Ian Agol after they received the Clay Research Award with Danny Calegari (not pictured).

The Clay Mathematics Institute (CMI) presents the Clay Research Award annually to recognize major advances in mathematical research. Recipients of the Clay Research Award are named as Clay Research Scholars, and receive flexible research support for a period of one year. They also receive the bronze sculpture "Figureight Knot Complement VII/CMI" by Helaman Ferguson. CMI gave two awards this year. The recipients are: Jean-Loup Waldspurger; Ian Agol, Danny Calegari, and David Gabai. Previous recipients of the award are Cliff Taubes and Claire Voisin (2008), Alex Eskin, Christopher Hacon and James McKernan and Michael Harris and Richard Taylor (2007), Manjul Bhargava and Nils Dencker (2005), Ben Green, Gérard Laumon and Bao-Châu Ngô (2004), Richard Hamilton and Terence Tao (2003), Oded Schramm and Manindra Agrawal (2002), Edward Witten and Stanislav Smirnov (2001), Alain Connes and Laurent Lafforgue (2000), and Andrew Wiles (1999).

Jean-Loup Waldspurger

For his work in p-adic harmonic analysis, particularly his contributions to the transfer conjecture and the fundamental lemma. This work, combined with that of others, makes it possible to finally resolve important, long-standing parts of the Langlands program.

Ian Agol, Danny Calegari and David Gabai

For their solutions of the Marden tameness conjecture, and, by implication through the work of Thurston and Canary, of the Ahlfors measure conjecture.

The Langlands program is a collection of conjectures and theorems that unify the theory of automorphic forms, relating it intimately to the main stream of number theory, with close relations to harmonic analysis on algebraic groups as well as arithmetic algebraic geometry. Since its origins in the winter of 1966-67, when it was laid out in a letter from Langlands to André Weil, it has served as the basis of much deep work, including applications to many famous problems in number theory, e.g., Artin's conjectures on L-functions, Fermat's last theorem, and the behavior of Hasse-Weil zeta functions.

The tameness conjecture asserts that a hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold (possibly with boundary). The Ahlfors conjecture asserts that the limit set of a finitely generated Kleinian group (i.e., the minimal invariant set on the Riemann sphere, which is the boundary at infinity of hyperbolic 3-space) has either full or zero measure, and in the former case the action of the group on it is ergodic.

Clay Research Awards

Transfer, the fundamental lemma, and the work of Waldspurger

by James Arthur

Recognizing Achievement

Jean-Loup Waldspurger

Automorphic forms are eigenfunctions of natural operators attached to reductive algebraic groups. Their eigenvalues are of great arithmetic significance. In fact, the information they contain is believed to represent a unifying force for large parts of number theory and arithmetic algebraic geometry.

The Langlands program is a collection of interlocking conjectures and theorems that govern the theory of automorphic forms. It explains in precise terms how this theory, with roots in harmonic analysis on algebraic groups, characterizes some of the deepest objects of arithmetic. There has been substantial progress in the Langlands program since its origins in a letter from Langlands to Weil in 1967. In particular, it has had applications to famous problems in number theory, including Artin's conjecture on L-functions, Fermat's last theorem, the Sato-Tate conjecture, and the behaviour of Hasse-Weil zeta functions. However, its deepest parts remain elusive.

At the center of the Langlands program is the principle of functoriality, a series of conjectural reciprocity laws among automorphic forms on different groups. There appears to be no direct way to prove it in any but the simplest of cases. One strategy for more general cases has been to compare trace formulas. The general trace formula for a reductive algebraic group G over a number F is a complex identity, which relates spectral and geometric objects. The spectral side contains the inaccessible data in automorphic forms to which the principle of functoriality applies. The geometric side is more explicit, but also more complicated. It is a sum of various kinds of integrals over spaces attached to G. The general idea is to compare the spectral data on different groups by establishing relations among the geometric terms in the corresponding trace formulas.

A serious obstruction for over thirty years has been the transfer of test functions between different groups. Given a smooth function of compact support for one group G, one tries to define a function on the second group by requiring that the orbital integrals on the geometric side of its trace formula match those of the first function. The problem is to show that these numbers really do represent the integrals of a smooth function of compact support. The transfer conjecture was formulated precisely by Langlands and Shelstad, and then later by Kottwitz and Shelstad for more general twisted

Clay Research Awards

James Arthur continued

groups. The conjecture included a family of explicit functions, called transfer factors, by which orbital integrals on one group would have to be multiplied in order to be orbital integrals on the other. Transfer factors are themselves a remarkable part of the story. They are natural if complicated objects, whose construction goes to the heart of class field theory.

A test function is defined on the adèlic group $G(\mathbb{A})$. At almost all p-adic places v of F, it is required to be the characteristic function of a maximal compact subgroup of $G(F_v)$. The fundamental lemma is a variant of the transfer conjecture. It asserts that the transfer of such a function must be of the same form, namely the characteristic function of a maximal compact subgroup of the F_v-points of the second group. The fundamental lemma appears at first to be a combinatorial problem, for the orbital integrals of characteristic functions reduce immediately to finite sums of terms that can be calculated explicitly. However, there are infinitely many orbital integrals to be treated, and as they vary, the number of terms in the associated finite sums increases without bound. Various elementary methods have been applied to the fundamental lemma over the years, but they have always met with at best limited success.

In the mid 1990s, Jean-Loup Waldspurger proved that the transfer conjecture would follow from the fundamental lemma. This was quite a surprise, for the fundamental lemma pertains to very special functions at certain p-adic places, while the transfer conjecture applies to general functions at all p-adic places. (The transfer problem for archimedean places had been solved earlier by Shelstad, using the work of Harish-Chandra. In fact, her solution served as a guide for the later construction of general transfer factors.) Waldspurger used global methods, specifically a simple version of the trace formula, to solve what was a local problem. In the past few years, he has also completed a far-reaching study of twisted harmonic analysis, which among other things, reduces the twisted transfer conjecture of Kottwitz and Shelstad to a twisted form of the fundamental lemma.

The breakthrough for the fundamental lemma was provided by Bao-Châo Ngô. He discovered a striking way to interpret the geometric side of the simple trace formula (or rather its analogue for a global field of positive characteristic). He observed that the entire geometric side could be expressed as a sum over the rational points of an arithmetic Hitchin fibration, the arithmetic analogue of a variety familiar from the theory of G-bundles on a Riemann surface. Earlier, Goresky, Kottwitz and MacPherson had discovered a geometric interpretation for the local terms in the simple trace formula (again for a global field of positive characteristic). Building on their work, and exploiting the interplay of local and global methods in ingenious ways, Ngô was at length able to establish a general form of the fundamental lemma.

Ngô's results actually apply to a p-adic Lie algebra of positive characteristic. They include a nonstandard fundamental lemma, which Waldspurger was led to conjecture in his study of twisted harmonic analysis. The theorem needed for the comparison of trace formulas applies to p-adic groups of characteristic 0. The link is provided by two separate results of Waldspurger. A reduction to a Lie algebra of characteristic 0 is included in his work on twisted harmonic analysis. The transition to Lie algebras of positive characteristic is a consequence of a different study, but one which is again based on methods of p-adic harmonic analysis. Another proof of this reduction was subsequently established by Cluckers, Hales, and Loeser, by completely different methods of motivic integration.

I should also mention a further generalization of the fundamental lemma which, like it or not, is also needed. It applies to the more exotic weighted orbital integrals, which occur in the general case of the trace formula. This has also been established recently. Exploiting the ideas of Ngô, with among other things, some remarkable new applications of intersection cohomology on which I am not qualified to comment, Chaudouard and Laumon have now proved a general form of the weighted fundamental lemma. It again applies only to a p-adic Lie algebra of positive characteristic. But Waldspurger, working at the same time, has also been able to extend his two theorems of reduction. The most general form of the fundamental lemma is therefore now available in all cases. Thanks to Waldspurger, this in turn implies the general form of the Kottwitz-Langlands-Shelstad transfer conjecture.

I have emphasized the role of transfer in the comparison of trace formulas. This is likely to lead to a classification of automorphic representations for many groups G, according to Langlands' conjectural theory of endoscopy. The fundamental lemma also has other important applications. For example, its proof fills a long-standing gap in the theory of Shimura varieties. Kottwitz had observed some years ago that the geometric terms in the arithmetic Lefschetz formula for a Shimura variety are twisted orbital integrals. The twisted fundamental lemma now allows a comparison of these terms with corresponding terms in the trace formula. This, in turn, leads to reciprocity laws between the arithmetic data in the cohomology of many such varieties with the spectral data in automorphic forms.

Let me conclude by saying that Waldspurger has made other major contributions to representation theory, which are quite independent of his pivotal role in the fundamental lemma and transfer. They include a large body of early work on the Shimura correspondence for modular forms that is still very influential, a classification of automorphic discrete spectra for general linear groups (with C. Moeglin), fundamental results on the homogeneity of p-adic characters and Shalika germs, a characterization of the stability properties of unipotent orbital integrals on p-adic classical groups, and a profound study for the group $SO(2n+1)$ of the representations of depth zero parametrized by Lusztig. As with all of Waldspurger's work, these contributions are marked by their depth, and by the application of Waldspurger's extraordinary mathematical power.

Clay Research Awards

Taming the field of hyperbolic 3-manifolds

by Jeffrey F. Brock

Jeffrey F. Brock

A hallmark of the modern study of 3-dimensional manifolds has been the role of geometry, or more specifically, *geometric structures*, in exploring their topology. W. Thurston's geometrization conjecture that each closed 3-manifold admits a decomposition into pieces each with a geometric structure is a powerful example of the force of geometry to render topological questions tractable, as its recent solution due to G. Perelman illustrates.

But last year's Clay Prize focuses on an earlier example of such a role for geometric structures on 3-manifolds, an example that beautifully draws together topology, geometry, and dynamics in low dimensions. The solution to Albert Marden's tameness conjecture represents a remarkable story of how a natural initial question can evolve and broaden in its implications and depth, cross-pollinating different subfields and disciplines along the way.

An open 3-dimensional manifold M is called *tame* if it is homeomorphic to the interior of a compact 3-manifold. In his efforts to prove Poincaré's conjecture that each simply connected closed 3-manifold M is homeomorphic to the 3-dimensional sphere, J.H.C. Whitehead discovered the first non-tame 3-manifold: a contractible 3-manifold topologically distinct from the open ball. This example led the way to numerous constructions of non-tame 3-manifolds with non-trivial fundamental group. Albert Marden, in his seminal investigation of the topological properties of 3-manifolds with a complete metric of constant negative curvature -1, the so-called *hyperbolic 3-manifolds*, formulated the following conjecture [Mar].

MARDEN'S TAMENESS CONJECTURE — *Each complete hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold.*

The conjecture, evidently easy to state, historically well motivated, and resilient, is made all the more remarkable by its profound importance and applications to disparate branches of mathematics. Last year's Clay Prize was awarded to Ian Agol, Danny Calegari, and David Gabai for two independent solutions of Marden's conjecture [Ag], [CG].

Its implications for

1. the Ahlfors measure conjecture for limit sets of finitely generated Kleinian groups, and
2. the classification of finitely generated Kleinian groups up to conjugacy,

lend it an unusual status at the intersection of topology, geometry, and dynamical systems

The network of implications and connections placing this conjecture in its modern framework owes a great debt to the work of W. Thurston, F. Bonahon, and R. Canary. We will attempt to elucidate this web of dependencies, describe the ramifications of the conjecture, now established, and to give a perspective on its proof.

A few words about Kleinian groups. At the center of the discussion is the notion of a Kleinian group Γ, namely, a discrete subroup of $\mathrm{PSL}_2(\mathbb{C})$, which plays alternatively the role of a group of Möbius transformations of $\widehat{\mathbb{C}}$ and a group of orientation-preserving isometries of hyperbolic 3-space $\mathbb{H}^3$ via the natural extension to the unit-ball model of $\mathbb{H}^3$. As the present discussion concerns groups with a manifold quotient $M = \mathbb{H}^3/\Gamma$, we will assume Γ is torsion free, and we will for simplicity omit any discussion of the case when Γ has parabolic elements, which correspond to embedded *cusp-regions* of M with standard topology.

The action of Γ on the Riemann sphere $\widehat{\mathbb{C}}$ determines a partition $\widehat{\mathbb{C}} = \Lambda \sqcup \Omega$ of the sphere into its *limit set* $\Lambda = \overline{\Gamma(0)} \cap \widehat{\mathbb{C}}$, the smallest closed Γ-invariant subset of $\widehat{\mathbb{C}}$ and its *domain of discontinuity* Ω, where Γ acts properly discontinuously by conformal homeomorphisms.

The quotient $(\mathbb{H}^3 \cup \Omega)/\Gamma$ gives a partial boundary for the complete hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, by adjoining Ω/Γ, the *conformal boundary* of M, a finite collection of finite-type Riemann surfaces by Ahlfors' *finiteness theorem* [Ah1].

The convex hull $CH(\Lambda)$ in hyperbolic space of the limit set Λ is the smallest hyperbolically convex set in $\mathbb{H}^3$ that contains Λ in its closure. As $CH(\Lambda)$ is Γ-invariant, its quotient $CH(\Lambda)/\Gamma$ is a geometrically preferred subset $C(M)$ of M, the *convex core* of M.

Ahlfors' conjecture. One of the more remarkable features of Marden's conjecture is its implication for conformal dynamical systems, established by Thurston, Bonahon, and Canary some years after its formulation.

AHLFORS' MEASURE-ZERO CONJECTURE — *If the limit set Λ of a finitely generated Kleinian group Γ is not all of $\widehat{\mathbb{C}}$, Λ has zero Lebesgue measure in $\widehat{\mathbb{C}}$.*

The conjecture was later expanded to include the expectation of ergodicity of the action of Γ on Λ if $\Lambda = \widehat{\mathbb{C}}$. Ahlfors' motivation rested on questions in the quasi-conformal deformation theory of Kleinian groups. For if each such limit set has measure zero, a non-trivial quasi-conformal deformation of a Kleinian group induces a non-trivial quasi-conformal deformation of its conformal boundary, guaranteeing that this deformation theory rests on a proper understanding of the Teichmüller theory of the conformal boundary.

Clay Research Awards

Jeffrey F. Brock continued

But it is perhaps lucky that Ahlfors did not have access to modern computer renderings of the limit sets of finitely generated Kleinian groups, as the conjecture might never have been made in the first place (see Figure 1).

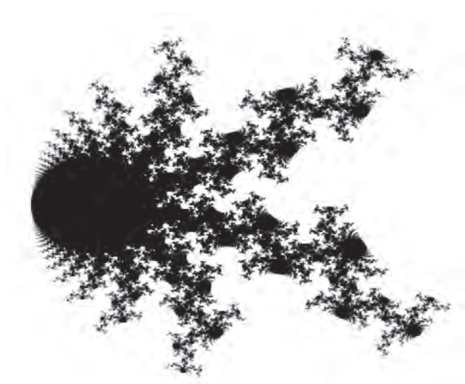


Figure 1.
A measure-zero limit set of a degenerate group.

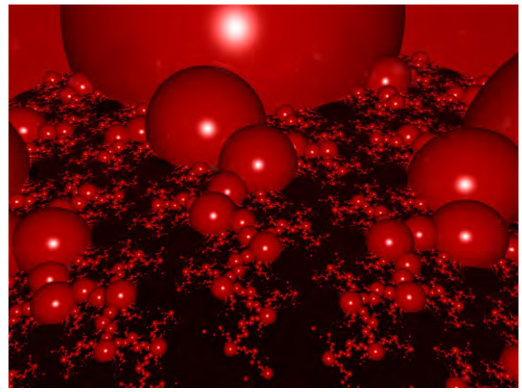


Figure 2. Inside the convex hull of the limit set.

Marden's conjecture. Marden set out to give a coherent description of the topological type of the quotient spaces $\mathbb{H}^3/\Gamma$ of finitely generated Kleinian groups. In the *geometrically finite* setting, when the convex core is assumed to have a finite volume unit neighborhood, his proof that the quotient manifold is homeomorphic to the interior of the compact manifold represented the first deep investigation of this topological question [Mar].

It was in this article that he formulated the tameness conjecture, predicting that such a simple topological description should exist for any finitely generated Kleinian group Γ. This conjecture would later be proven in more general cases and in turn be reinterpreted to serve as a centerpiece of Thurston's conjectural classification of finitely generated Kleinian groups. The conjecture grows out of the *core theorem* of Peter Scott [Sco] that each 3-manifold with finitely generated fundamental group admits a compact submanifold whose inclusion is a homotopy equivalence. In the setting of a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, then, finite generation of Γ guarantees the existence of a decomposition into a compact submanifold and a finite collection of complementary pieces, termed *ends* of the manifold (more properly, each piece is a *neighborhood of an end*, of M, depending on the choice of core).

To prove Marden's conjecture amounts to proving a choice of compact core $\mathscr{M}$ exists so that each end $\mathscr{E}$ is a product $S \times \mathbb{R}^+$ where S is a component of the boundary $\partial\mathscr{M}$. In the geometrically finite setting, a nearest point retraction map from the conformal boundary Ω/Γ provided a natural product structure for the ends of $\mathbb{H}^3/\Gamma$, which later came to be called *geometrically finite ends*.

Wild ends and tame ends. It is instructive to review what can go wrong in general. We briefly review Whitehead's construction: consider S^3 decomposed into solid tori U and V along a common torus boundary component T. Let $h\colon V \to V$ denote the pictured embedding of V

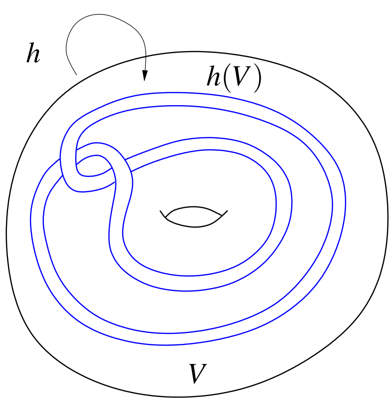


Figure 3. Whitehead's embedding.

into itself: $\mathrm{int}(V) \setminus h(V)$ is then the complement of the Whitehead link in S^3. A meridian J of V is then homotopically essential in $V \setminus h(V)$ but bounds a disk in $S^3 \setminus h(V) = U \cup (V \setminus h(V))$.

Then $J, h(J), h^2(J), \ldots, h^n(J)$ denote loops that are homotopically distinct and non-trivial in $V \setminus h^n(V)$, all of which become trivial in $S^3 \setminus h^n(V)$.

Letting $X_\infty = \cap_n h^n(V)$, the *Whitehead manifold* $S^3 \setminus X_\infty$ is a simply connected, indeed contractible 3-manifold with the property that the removal of a compact submanifold V leaves a 3-manifold with infinitely generated fundamental group (by an application of Van Kampen's theorem). Such a manifold cannot be homeomorphic to $\mathrm{int}(B^3)$.

Note, however, that by the Cartan-Hadamard theorem, there is a unique simply connected complete Riemannian manifold of constant curvature -1, namely, hyperbolic 3-space $\mathbb{H}^3$. So the assumption of negative curvature tames the topology in this case.

What about lifting the assumption of simple connectivity? In a variant of Whitehead's construction due to Tucker [Tck], one replaces U and V with a handlebodies of the same genus, and $h\colon V \to V$ becomes a knotted embedding of V into itself (h is homotopic but not isotopic to the identity). The result is an open 3-manifold that is *exhausted by compact cores*, but for which the complement of U has infinitely generated fundamental group. The possibility

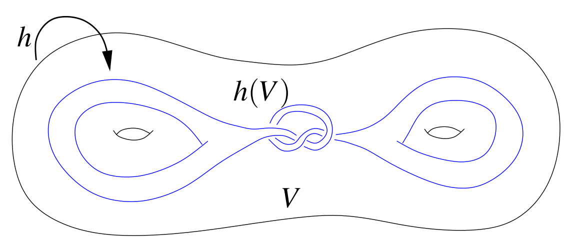


Figure 4. A knotted handlebody.

that such a 3-manifold might admit a complete hyperbolic structure was later ruled out by a result of Souto [Sou], but examples of sequences of embeddings with unbounded genus and more

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and more complicated knotting remained out of reach until the solution of the full conjecture.

Geometric tameness. Though Ahlfors established his measure-zero conjecture for geometrically finite Kleinian groups [Ah2], W. Thurston described an extension of Ahlfors' theorem in his Princeton Lecture Notes *Geometry and Topology of Three-Manifolds* [Th1], employing very directly the internal geometry of the quotient hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ where Γ is a finitely generated torsion-free Kleinian group.

A centerpiece of Ahlfors' argument involves the natural harmonic extension of the characteristic function of a measurable set on the Riemann sphere $\widehat{\mathbb{C}}$ to a harmonic function $\tilde{h}$ on hyperbolic space $\mathbb{H}^3$, the solution to the *Dirichlet problem*. If the set is invariant by the action of the Kleinian group Γ, its characteristic function and hence its extension are as well, and $\tilde{h}$ descends in the quotient to a harmonic function h on the hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$. Harmonicity of h guarantees that its gradient flow is volume preserving.

A geometric condition introduced by Thurston that generalizes geometric finiteness, termed *geometric tameness*, ensures that no positive measure set of flow lines can exit an end of the convex core $C(M)$. This gives a maximum principle for non-constant harmonic functions on $C(M)$, showing that no measurable invariant subset of Λ can have a point of Lebesgue density unless it is all of $\widehat{\mathbb{C}}$.

Briefly, the assumption of geometric tameness for the end $\mathscr{E}$ of $C(M)$ provides for intrinsically hyperbolic surfaces X_n exiting the end $\mathscr{E}$, each homotopic to the boundary S of $\mathscr{E}$; the geometry of X_n guarantees a bounded-diameter statement that controls the volume swept out by a unit neighborhood of each X_n, which in turn serves to limit the measure of flow lines of ∇h crossing X_n. The manifold $\mathbb{H}^3/\Gamma$ is geometrically tame if all its ends are geometrically finite (asymptotic to a component of the conformal boundary) or geometrically tame.

Topological and geometric tameness. Thurston established the topological implications of geometric tameness in many settings in his Notes [Th1]. He used the surfaces exiting the end to build a topological product structure for the end, proving *topological tameness* in these settings. A key role is played by sequences of closed geodesics $\{\gamma_n\}$ that exit $\mathscr{E}$ that can be made *simple* (non self-intersecting) when realized as curves on the boundary S. Such geodesics serve to anchor the surfaces X_n and force them to infinity.

Thurston's geometric condition, moreover, gave rise to a new invariant of the geometry of a geometrically tame end, namely, its *ending lamination*, a kind of limit of $\{\gamma_n\}$. A posteriori, the ending lamination $\nu(E)$ encodes the limiting combinatorial picture of bounded length closed geodesics that exit the end E. It is only clearly well defined, however, when the end is assumed geometrically tame.

In ensuing years, Bonahon showed that one can always find closed geodesics $\{\gamma_n\}$ that may fail the simplicity assumption above, and showed how to use their limit to find new geodesics $\{\gamma'_n\}$ that do satisfy the simplicity condition, provided that $S \hookrightarrow M$ is injective on the level of fundamental group [Bon]. Geometric tameness followed in this case. Later, Canary showed how to relax Bonahon's hypotheses in such a way to show that any topologically tame hyperbolic 3-manifold is *geometrically tame*, thereby reducing Ahlfors' original conjecture to Marden's [Can1]. He also showed how to generalize Thurston's argument to build topological product structures for general geometrically tame ends, thereby showing the equivalence of topological and geometric tameness.

The proof. A complete description of the proof is naturally out of the scope an article such as this one. Essentially, the goal rested in trying to find the appropriate replacement for the simple closed geodesics $\{\gamma_n\}$ exiting an end $\mathscr{E}$ that is not geometrically finite as a method to produce exiting surfaces X_n in the same homotopy class. It is interesting to note both how geometry and topology played key roles:

1. The original argument of Bonahon guarantees the existence of closed geodesics $\{\gamma_n\}$ exiting the end $\mathscr{E}$, not necessarily homotopic to simple curves on S.
2. A topological innovation called an *end-reduction* allows one to find surfaces in the end that lie outside or "engulf" an arbitrary finite subcollection of these geodesics.
3. Such surfaces can be pulled tight to geometric surfaces in the complement of the geodesics they enclose: the sets of geodesics are said to be "shrinkwrapped" by a geometric $(CAT(-1))$ surface, and infinitely many such lie in the same homotopy class.
4. A bounded diameter lemma together with a homology argument guarantees these surfaces exit the end, and then standard techniques produce the desired product structure as in the geometrically tame setting.

A critical topological insight was to notice the efficacy of technology on analyzing the ends of open 3-manifolds due to M. Brin and T. Thickstun, and R. Myers. Indeed, their *end reductions* provide the key facts from 3-manifold topology to guarantee the usefulness of the exiting geodesics — one can imagine that they provide cellophane that is used in the shrinkwrapping. In effect, the shrinkwapped surfaces Z_n replace the surfaces assumed in geometric tameness, and the exiting geodesics γ_n serve to show the surfaces Z_n exit the end $\mathscr{E}$.

We remark that we have focused on the solution presented by Calegari and Gabai, in which shrinkwrapping is accomplished using minimal surfaces, but a more recent treatment due to T. Soma employs standard polyhedral techniques to obtain the same result [So].

The Classification theorem. Much of Ahlfors' original motivation for the measure-zero conjecture was obviated in practice by Sullivan's rigidity theorem [Sul], which guaranteed the absence of deformations supported on the limit set alluded to previously. In more recent years, it

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was the progress toward and ultimate solution to the *ending lamination conjecture* of Thurston due to Minsky [Min] and concluded by work of this author with Canary and Minsky [BCM2, BCM1] that brought renewed attention to the tameness conjecture.

The ending lamination conjecture predicts that each geometrically tame hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is determined up to isometry by its homeomorphism type, its cusps (regions corresponding to *parabolic* elements of Γ), and its *end invariant* $\nu(M)$ consisting of the conformal structures on Ω/Γ and the ending laminations $\{\nu(\mathscr{E})\}$ for each geometrically tame end of $C(M)$. But in the intervening years, the aforementioned work of Bonahon and Canary showed the *topological* tameness of M to be the complementary conjecture for a complete classification theorem.

THE CLASSIFICATION THEOREM — *Each complete hyperbolic 3-manifold with finitely generated fundamental group is determined up to isometry by its topology, its cusps, and its end invariant.*

The theorem, which formally combines the tameness theorem and the ending lamination theorem, sets to rest what has been perhaps the central motivating conjectural question in finitely generated Kleinian groups. It is notable however, that the output is richer than simply a classification: the method of proof of the ending lamination theorem produces a combinatorial model for the ends M directly from the end-invariant data $\nu(M)$, and thus a uniform picture of its hyperbolic metric up to bi-Lipschitz equivalence. That any finitely generated Kleinian group can now be understood so concretely provides new methods to study the internal geometry of the full spectrum of hyperbolic 3-manifolds, their deformation spaces, and how their topological, analytic, and geometric invariants interrelate.

Implications. In his seminal Bulletin article [Th2], Thurston's list of twenty-four problems and questions set the stage for the next thirty years of activity in the geometry and topology of 3-manifolds, and in particular the fields of Kleinian groups and deformation theory of hyperbolic 3-manifolds. (It is notable that Thurston's celebrated *geometrization conjecture* is question 1 on this list. To reflect on how far the field of geometric structures on 3-manifolds has progressed in the last ten years is, once again, beyond the scope of this article, but we simply state, for emphasis, questions in Thurston's list in which the solution to tameness plays a central role.

1. AHLFORS' MEASURE CONJECTURE. Thurston's harmonic flow argument together with Canary's theorem that topologically tame implies geometrically tame guarantee that the limit set Λ of a finitely generated Kleinian group has zero or full measure, and if full, the action of Γ is ergodic on Λ.

2. THE CLASSIFICATION OF FINITELY GENERATED KLEINIAN GROUPS. Because ending laminations are only defined for tame ends, the proof of the ending lamination conjecture [BCM2, BCM1] requires

tameness to apply to all finitely generated Kleinian groups.

3. THE DENSITY CONJECTURE. After work of Namazi and Souto [NS], each candidate end-invariant can be realized in a limit of geometrically finite manifolds, and hence the limit is isometric to a given (necessarily tame) manifold by the ending lamination conjecture. This resolves the *Density Conjecture* of Bers, Sullivan, and Thurston (independently resolved by [BS]).

4. THE MODEL MANIFOLD CONJECTURE. A combinatorial bi-Lipschitz model for the ends of hyperbolic 3-manifolds with finitely generated fundamental group, conjectured by Thurston, arises directly from the ending lamination in the proof of the ending lamination conjecture, and is thus only operative in full generality after tameness.

There are numerous other deep implications of tameness and the model manifold theorem for the geometry, topology, and dynamics of finitely generated Kleinian groups and their associated hyperbolic 3-manifolds. The ergodicity of the geodesic flow on the unit tangent bundle, Simon's tameness conjecture for covers of compact manifolds, the recently claimed local-connectivity theorem for limit sets of finitely generated Kleinian groups [Mj], and the enumeration of components of the deformation space of a Kleinian group [BCM2], are just a few other major examples. For a survey of these and many other applications see [Can2].

The story of the symbiosis between geometry and topology in 3-dimensions continues to be written, and more and more precise connections between geometric and topological invariants for 3-manifolds emerge with ever-increasing frequency. The tameness theorem taken together with the model manifold theorem guarantees that for each finitely generated Kleinian group Γ, the hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ can be modeled in a combinatorial way on its ending laminations. Such a combinatorial structure provides many new methods and tools, and indeed new questions for investigation in the study of 3-manifolds.

An apocryphal story has it that Ahlfors submitted a one-line grant proposal late in his career containing the single sentence: "I will continue to try to understand the work of Thurston." No doubt he would be gratified to see how the cumulative efforts of so many mathematicians have culminated in such a rich narrative, intertwining the solution to his own conjecture with those of Marden, Thurston, Bers, and Sullivan, and how so many fundamental questions in the geometry, topology, and dynamics of Kleinian groups have been set to rest.

Acknowledgtments. The author thanks Juan Souto and Dick Canary for comments on a draft version of this article. Computer programs of Curt McMullen were employed to produce the images of limit sets and their convex hulls.

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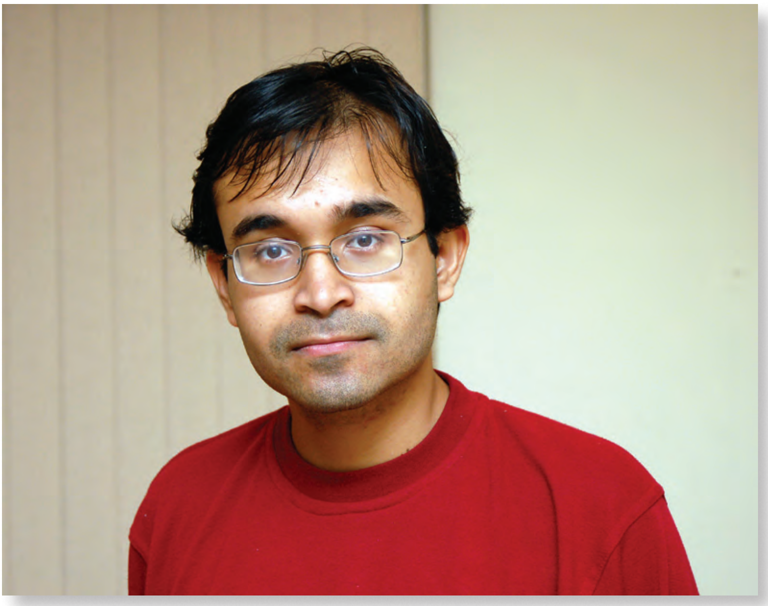
Summary of 2009 Research Activities



Sucharit Sarkar

The Activities of CMI researchers and research programs are sketched below. Researchers and programs are selected by the Scientific Advisory Board (see inside front cover).

Clay Research Fellows

Sucharit Sarkar, born in Calcutta, India, received his PhD from Princeton University in 2009 under the guidance of Zoltan Szabo. His research area is in low dimensional topology. His dissertation addressed topics in Heegaard Floer homology for 3-manifolds and knots inside 3-manifolds. He began his five-year appointment in July of 2009.

Sucharit Sarkar joined CMI's current group of research fellows Mohammed Abouzaid (MIT), Spyros Alexakis (U of Toronto), Artur Avila (IMPA Brazil), Maria Chudnovsky (Columbia University), Soren Galatius (Stanford University), Adrian Ioana (Caltech), Bo'az Klartag (Princeton University), Ciprian Manolescu (Columbia University), Davesh Maulik (Columbia University), Maryam Mirzakhani (Princeton University), Sophie Morel (Institute for Advanced Study), Samuel Payne (Stanford University), David Speyer (MIT), Teruyoshi Yoshida (Harvard University), and Xinyi Yuan (Institute for Advanced Study).

Research Scholars

Bryna Kra (Northwestern)
January 5, 2009–June 4, 2009
at University of Marne la Vallée in Paris.

Fernando Rodriguez-Villegas (U of Texas at Austin)
September 1, 2008–August 31, 2009
at Oxford University.

Richard Schwartz (Brown University)
February 1, 2009–March 31, 2009
at Caltech.

Kasra Rafi (U of Oklahoma)
January 5, 2009–June 5, 2009
at the University of Chicago.

Senior Scholars

Claire Voisin (IHES)
February 1, 2009–February 28, 2009
at the MSRI program on Algebraic Geometry.

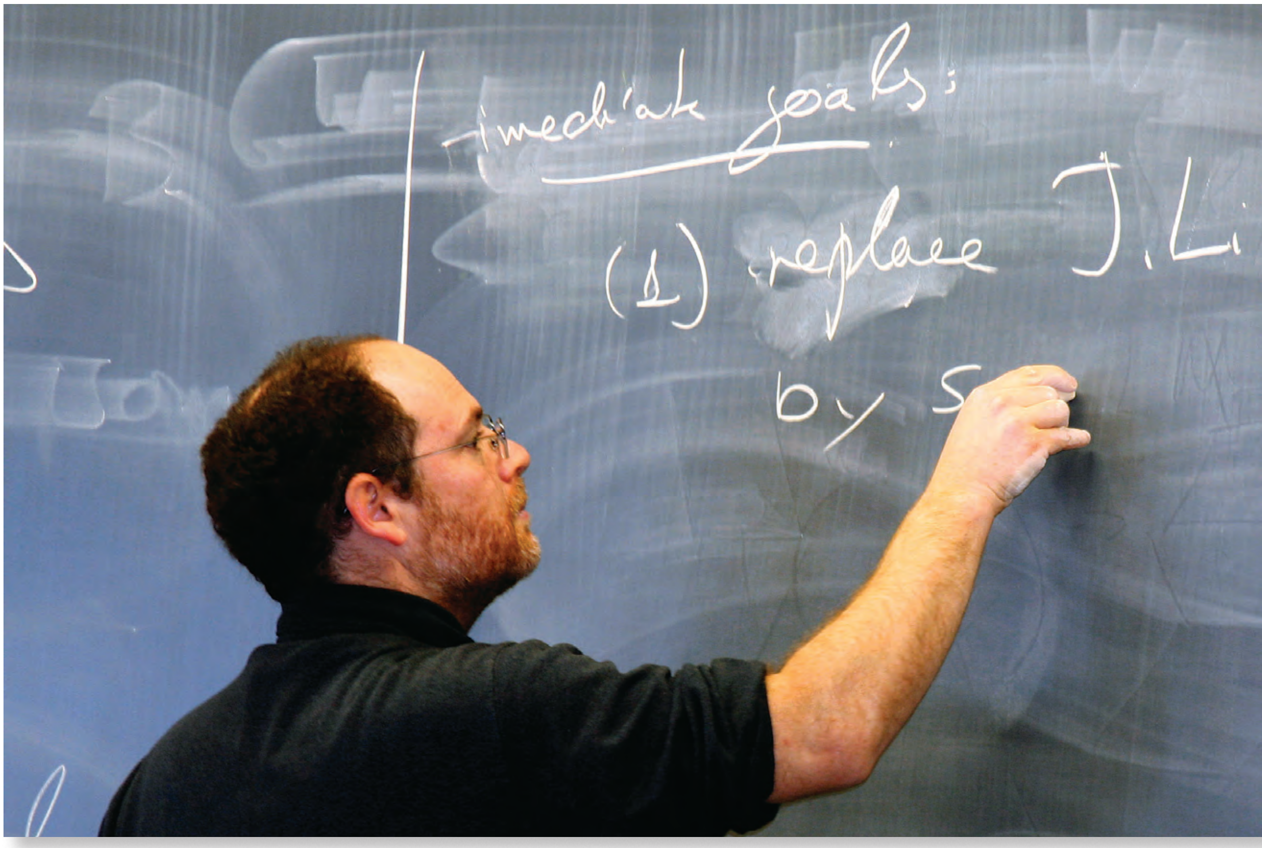
Christopher Hacon (U of Utah) and
Rahul Panharipande (UC, Berkeley)
April 1, 2009–May 16, 2009
at the MSRI program on Algebraic Geometry.

Benedict Gross (Harvard University)
June 28, 2009–July 18, 2009.
at the PCMI program on "The Arithmetic of L-functions,"
part of the PCMI Clay Senior-Scholar-in-Residence program.

Clifford Taubes (Harvard University)
August 1, 2009–December 31, 2009.
at MSRI's program on "Symplectic and Contact Geometry and Topology."

John Tate (University of Texas at Austin, Harvard University)
June 28, 2009–July 18, 2009
at the PCMI program on "The Arithmetic of L-functions,"
part of the PCMI Clay Senior-Scholar-in-Residence program.

Summary of 2009 Research Activities continued



Dan Abramovich, CMI Workshop - Geometry and Physics of the Landau-Ginzburg Model

LiftOff Fellows

CMI appointed seven LiftOff Fellows for the summer of 2009. Clay LiftOff Fellows are recent PhD recipients who receive one month of summer salary and travel funds during the summer following their graduation.

David Anderson
Jonah Blasiak
Victor Lie
Grigor Sargysan
Andrew Snowden
Melanie Wood
Xinwen Zhu

Research Programs organized and supported by CMI

January 12-16. CMI Workshop - Geometry and Physics of the Landau-Ginzburg Model, Cambridge.

March 2-5. Clay Lectures on Mathematics, Kyoto, Japan.

March 17-20. IV International Symposium on Nonlinear Equations and Free Boundary Problems, Buenos Aires, Argentina.

April 5. Singularities @ MIT, Cambridge.

April 20-22. Geometry and Physics: Atiyah80, Edinburgh, UK.

Researchers, Workshops & Conferences

May 4-5. Clay Research Conference, Harvard.

May 4-8. Power of Analysis, Princeton.

June 2-30. Thematic Program on Probabilistic Methods in Functional Analysis, CRM Montreal, Canada.

June 15-July 10. CMI Summer School - Galois Representations, Hawai'i.

June 22-26. Topology of Algebraic Varieties Conference, Jaca, Spain.

June 24-30. Geometry and Functional Analysis in honor of Vitali Milman's 70th birthday, Tel Aviv, Israel.

July 6-10. Journées de Géometrie Arithmétique de Rennes, France.

July 23-October 7. Clay-Mahler Lecture Tour, Australia.

July 20-24. Dynamical Numbers: Interplay Between Dynamical Systems and Number Theory, Bonn, Germany.

October 19-22. CMI Workshop - Geometry of Outer Space, Cambridge.

December 1-5. CMI Workshop - Sage Days 18, Cambridge.

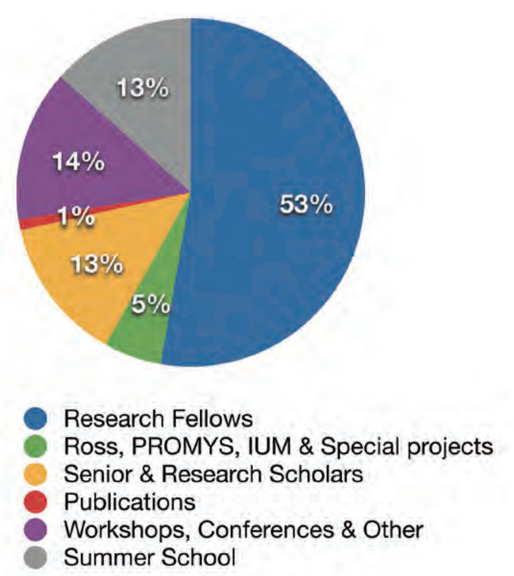
December 15-23. Geometry and Probability: the mathematics of Oded Schramm, Jerusalem, Israel.

Program Allocation

Estimated number of persons supported by CMI in selected scientific programs for calendar year 2009:

Research Fellows, Research Awardees, Senior Scholars, Research Scholars, LiftOff Fellows	36
Summer School Participants and Faculty	95
PROMYS/Ross Participants and Faculty	22
CMI Workshops	45
Participants attending Conferences and Joint Programs	> 5000
Independent University of Moscow (IUM)	80

Research Expenses for Fiscal Year 2009



Twenty Years of PROMYS
The Program in Mathematics for Young Scientists

Alumni gather at the Twenty Years of PROMYS Celebration at Boston University in July, 2009.

In July 2009, almost 200 friends and colleagues gathered from around the country and from as far away as Europe and Asia for a weekend at Boston University to celebrate a milestone: Twenty Years of PROMYS, the Program in Mathematics for Young Scientists. Alumni representing all Twenty years of the program reconnected with old friends as they reminisced and shared stories from their time in the program. They attended mathematical lectures by eminent fellow alumni on topics ranging from sphere packing (Henry Cohn of Microsoft Research), through cryptography (David Jao of University of Waterloo), integral representations of quadratic forms (Jonathan Hanke of University of Georgia), logic (Cameron Freer of MIT), discrete mathematics (Blair Dowling Sullivan of Oak Ridge National Laboratories), and mathematics education (Dev Sinha of University of Oregon), to a discussion of what went wrong in the recent international financial crisis (Tom Brennan of Northwestern University). There was much discussion of the history of the program as well as of plans and ambitions for the future. Explicit and implicit throughout the Celebration was how important, often pivotal, PROMYS has been to the intellectual and career development of alumni. At PROMYS, they lived and worked for six weeks as mathematicians and scientists; and they believe that this had a long-term positive impact on their lives.

PROMYS is an intensive and challenging six-week immersive program in mathematics that has been held at Boston University for each of the last twenty-one summers. The program was founded in 1989 by a group of mathematicians who were themselves alumni of another summer program that had been pivotal to their own intellectual and career development: the famous Ross Program at Ohio State University, which continues to run strongly more than Fifty years after its inception. Like the Ross Program, PROMYS emphasizes creative mathematical investigation that includes experimentation, rigorous proof, and precise use of language, as well as the practice of the techniques of abstraction and generalization.

Starting in 1989, PROMYS was supported by the National Science Foundation's Young Scholars Program (YSP). But following the demise of NSF's YSP in 1997, the program struggled financially for several years until the Clay Mathematics Institute and other leading American mathematical institutions (notably AMS and NSA) came forward to offer aid and support. In 1999, the CMI/PROMYS partnership was founded with the specific goal of extending and deepening the advanced features of PROMYS. Since then the partnership has grown and thrived as new dimensions have been added to the traditional PROMYS experience, including research opportunities for high school students guided by research mathematicians, and advanced seminars for counselors and students. Student research projects of recent years have included investigation into themes such as hyperbolic geodesics and continued fractions, tropical algebraic geometry, Gelfand-Tsetlin patterns, finiteness theorems for integral quadratic forms, orbital systems, and the Artin-Hasse exponential series. Problems have been proposed and mentored by many mathematicians including Ben Brubaker, Henry Cohn, Keith Conrad, Paul Gunnells, Jonathan Hanke, Kiran Kedlaya, Jonathan Lubin, Dev Sinha, and others. At the end of each summer, students present their research to the entire PROMYS community.

About seventy-five mathematically gifted and highly motivated high school students join PROMYS each summer. Of these, about twenty are returning students who come back to participate in the more advanced activities of the program. They join a staff of approximately eighteen counselors, outstanding undergraduates selected from top mathematics departments around the country often themselves PROMYS alumni. A major aspect of the PROMYS experience is the extended mathematical learning community formed by the first-year students, the returning students, the counselors, the faculty, and the visiting mentors and speakers—all of whom are creatively, intensely, and collaboratively focused on mathematical exploration. At the end of each summer all the documented elements of that summer's program—the problem sets and notes from courses and seminars, as well as student research papers summarizing results—are collated into a bound volume which is then presented to all members of the PROMYS community. One of the central visual foci at the July Celebration was a long table laid out with twenty such annual testaments to creative intellectual struggle and sustained mental effort.

The Twenty Years of PROMYS Celebration in July was the first-ever organized PROMYS alumni gathering. Encouraged by the overwhelmingly enthusiastic response from alumni who attended the Celebration (and by many eager but unable to attend), PROMYS began reaching out to the community to reestablish contact with those alumni with whom contact had been lost. Nine months later, the results are stunning—the program is now in touch with over 95% of its 1,144 alumni. One snapshot of data gathered: of the 80% of alumni for whom the program has seemingly up-to-date educational data, 43% of those old enough have completed, or are working on, a PhD in a science, technology, engineering, or mathematics (STEM) field. Of the forty-siz comparable alumni who have attended PROMYS more than three times, 78% have completed, or are working on, a STEM PhD, almost all of which are in mathematics. It is enlightening to look back and exciting to look forward. The Celebration has acted as a springboard for the alumni to extend and deepen their involvement with the program, and this can only strengthen PROMYS in the years ahead.

Margy Baruch, Glenn Stevens (Director), David Fried, and Steve Rosenberg: founders and current faculty of PROMYS.

Interview with Research Fellow Artur Avila

"I have worked on quite distinct areas: one-dimensional dynamics, ergodic Schrödinger operators, Teichmüller flow and interval exchange transformations, volume preserving and partially hyperbolic maps.

Artur Avila (b. 1979) received his PhD in 2001 under the direction of Welington de Melo at the Instituto Nacional de Matemática Pura e Aplicada (IMPA), in Rio de Janeiro, Brazil. His thesis, "Bifurcations of unimodal maps: the topological and metric picture," generalized the regular or stochastic dichotomy from the quadratic family to any non-trivial family of real analytic unimodal maps. Since then he has made numerous outstanding contributions to one-dimensional and holomorphic dynamics, spectral theory of the Schrödinger operator, and ergodic theory of interval exchange transformations and the associated Teichmüller flow. Avila is a Chargé de Recherche at the Centre Nationale de Recherche (CNRS). Below is an interview with Avila.

What first drew you to mathematics? What are some of your earliest memories of mathematics?

My father, who came from Amazonas and was not able to start school until age fourteen or so, was enthusiastic about exposing me early to basic education, including mathematics. At an early age, I was attracted by big numbers (like the speed of light in meters/second), and thought that multiplying trillions was a great topic for conversation (somehow the other kids did not agree).

Could you talk about your mathematical education? What experiences and people were especially influential?

I studied in an OK school, but for several years I was mostly reading on my own. At fifth grade the program underwent a phase transition. They were impressed with modern mathematics, so I guess they thought we should learn about set theory and various axioms. Unfortunately, I do not think I was quite ready to appreciate the notion of equivalence class. Infinite sets were fun and it was nice to learn the Cantor-Bernstein-Schröder theorem, but I do not really consider the composition of homotheties to be more primitive than the multiplication of real numbers (even if properly introduced with Dedekind cuts). Although some teachers did not understand what they were talking about, one of them was very charismatic (exam question: if x belongs to the empty set, is x an elephant?). This charismatic teacher told me about the existence of mathematical Olympiads, and there I went. Suddenly, I was faced with much more concrete problems, and after the initial shock, it became my main occupation until the middle of the tenth grade. In Brazil high school runs up to eleventh grade, or at least used to when I started at IMPA.

Did you have a mentor? Who helped you develop your interest in mathematics, and how?

Since I went to IMPA while still in high school, it was very important for me to be guided carefully. Elon Lages Lima was my mentor during my first two years there, when I was learning the foundations. I would frequently go to his office to discuss whatever I was getting excited about in the various courses, and he would give me his own perspective. As I was getting closer to starting the PhD, I started discussing mathematics more with Welington de Melo, who later on would become my advisor. He was a big influence, particularly on my mathematical taste. A bit later on, I got in touch with Misha Lyubich; his way of thinking made a big impression on me. Many years later we still collaborate on long projects that always seem to take years to mature...

After my PhD I spent two years in Collège de France with Jean-Christophe Yoccoz, who has a very different style. It was great to be constantly exposed to his way of doing math.

"After my PhD I spent two years in Collége de France with Jean-Christophe Yoccoz, who has a very different style. It was great to be constantly exposed to his way of doing math.

You were educated in Brazil. Could you comment on the differences in mathematical education there and in the US?

I have very little contact with the US system, so I will just comment on my experience with the Brazilian system. In my case an important role was played by mathematical olympiads: it was through them that I found out about IMPA. Such a special place, right in my home city! Rules there are not strict, so it was possible to follow graduate classes even before finishing high school. Since the quality of undergraduate education in math is not very high in Brazil, in practice the program starts from scratch and does not assume anything was learned beyond high school.

IMPA is of course a big attractor for the best students in Latin America, and many of my classmates went on to become very good mathematicians (and some of them coauthors).

What attracted you to the particular problems you have studied?

It depends a lot on my own understanding of the field. After getting interested in a new topic, I tend to get attracted to conjectures which seem to be at least vaguely connected with my previous work. For instance, with quasiperiodic Schrödinger operators, the link was renormalization. With Teichmüller flow, it was probabilistic parameter exclusion.

This outsider perspective tends to get replaced by an insider one with time, and eventually one gets one's own ideas of what should be done in the field. Then it is possible to just set some attractive but distant goal and concentrate on the bunch of problems that must be solved along the way.

Can you describe your research in accessible terms? Does it have applications to other areas?

I have worked on quite distinct areas: one-dimensional dynamics, ergodic Schrödinger operators, Teichmüller flow and interval exchange transformations, volume preserving, and partially hyperbolic maps. If one tries (perhaps too much) to find some common ground among the largest possible number of works, one could say that it often involves a bit of a conflict between order and chaos.

Artur Avila interview continued

> *Collaboration is a very efficient way to get into a new topic. I also like the obligation to communicate: it may be only at the moment one tries to verbalize thoughts that something clicks.*

For instance, the Teichmüller flow can be understood as a renormalization operator for translation flows: while the dynamics of the Teichmüller flow is very chaotic, translation flows are relatively ordered. Chaos brings in the power of probabilistic analysis, while the details of "relative order" may be incredibly messy, so what one can do is to see how the dynamics of the renormalization operator reflects on that of translation flows.

Sometimes what one wants is to prove a dichotomy order or chaos in a class of dynamics. An example is the basic measure-theoretical dichotomies, regular or stochastic for unimodal maps and reducibility or nonuniform hyperbolicity dichotomy for one-frequency SL(2,\R) cocycles. I am looking right now at a spectral version of this dichotomy, for the associated Schrödinger operators: trying to break the spectrum in two parts, one absolutely continuous, another localized.

Some of the analysis of higher dimensional dynamics also fits somewhat in this philosophy: one may start from a system with some hyperbolic directions and some neutral directions and find out that the only way not to create new non-zero Lyapunov exponents is to keep the ordered behavior in the central.

As for the second part of the question, I feel my work already involves a bit of applying at least part of the ideology of one of those areas (say, one-dimensional dynamics) to another (say, quasiperiodic Schrödinger operators).

What research problems and areas are you likely to explore in the future?

My impression is that it is difficult to predict one's research focus beyond a relatively short horizon. Slightly more than one year ago, I had no idea that my dream problems about one-frequency Schrödinger operators would be within reach. Then a little miracle happened and everything opened up. So right now I am making quite an effort trying to work it all out.

However, at the general level, I do have an intention to try to move further in some natural directions. I have played with aspects of the dynamics of group actions, so it would be sensible to try to expand my work there. On the other hand, since there is already quite a bit of probability on my work, it could also be interesting to try to work on certain stochastic processes.

Could you comment on collaboration versus solo work as a research style? Are certain kinds of problems better suited to collaboration?

My collaborations have been very diverse, ranging from basically a smooth dialogue (two or more people together in front of the blackboard, or in a chat window, for hours) to long stretches of essentially individual work interspersed with quick discussions.

Collaboration is also a very efficient way to get into a new topic. I do not read much and prefer to learn from mathematical discussions. I also like the obligation to communicate: it may be only at the moment one tries to verbalize thoughts that something clicks.

For all those reasons, I think the real question is why some problems do not seem to benefit from collaboration. But it does happen.

Regarding individual work versus collaboration, what do you find most rewarding or productive?

It is much more fun to collaborate. But usually most of the work is done while thinking obsessively night after night, by oneself.

How has the Clay Fellowship made a difference for you?

At my CNRS job in Paris, I already had quite a bit of freedom (no teaching duties, flexibility to travel). The Clay Fellowship gave me the opportunity to change my base to wherever I wanted. For five years I had been in the French system, and I already intended to spend some time at IMPA. So I did that. Initially I planned to stay for one year, but it was going so well that I extended it to the three years of my fellowship (while traveling often to the US and Europe). I found out I work somewhat differently when I am in Rio than when I am in Paris, and my impression is that the best is to try to combine both: going forward I will be splitting my time between both places.

What advice would you give to young people starting out in math (i.e., high school students and young researchers)?

There are so many different ways one can approach mathematics that I find it difficult to give practical advice that is not artificially constraining. Maybe this in itself is worthwhile to know: there is no one style of doing math that is a priori better than all others.

What advice would you give lay persons who would like to know more about mathematics—what it is, what its role in our society has been and is, etc.? What should they read? How should they proceed?

If the goal is to know what mathematicians do, a starter would be the article of Thurston, "On proof and progress in mathematics."

How do you think mathematics benefits culture and society?

Looking backwards, the largest impact must be through physics. How current mathematics will benefit society is harder to predict. Obviously through things like computer science, but it would seem too imprudent to discard a possible increasing mathematization of biology, for instance. Going further, it seems natural to me that a qualitative understanding of dynamical systems should benefit even those fields which are less amenable to precise quantification, like economics.

Please tell us about things you enjoy when not doing mathematics.

Living (part-time) in Paris, I try to take the time to enjoy my meals (usually with some wine). This may involve going to restaurants, which is often also a good opportunity to get together with friends. But since it can get tiresome to eat out too much (and while traveling I may find myself in some place with more restricted options), my girlfriend and I have started to work on the diversity of dishes that we can cook at home.

Besides this, I would say that I like old movies—I usually end up seeing them on DVDs.

> *"There are so many different ways one can approach mathematics that I find it difficult to give practical advice that is not artificially constraining. Maybe this in itself is worthwhile to know: there is no one style of doing math that is a priori better than all others.*

I Spy With My Little Eye

Mathematical Visualization and the Animation of Singularities in the Film *Zeroset* by Herwig Hauser

“*When I encountered resolution of singularities in the nineties, skepticism kept me from thoroughly entering the subject for a long time.*

Herwig Hauser is Professor of Mathematics at the University of Vienna in Austria, specializing in algebraic geometry. Among other things, he works in singularity theory, with a recent emphasis on the resolution of singularities in zero and prime characteristic. He is the author of the movie “ZEROSET – I spy with my little eye,” presented at the ICM 2006 in Madrid, and of a series of expository works on algebraic surfaces and singularities. The fictitious interview below took place in the Tyrolean Alps in August 2009.[1]

You have recently been working on the resolution of singularities in positive characteristic, a still-unsolved problem that is very algebraic in nature. The proof in characteristic zero already avoids a great deal of the geometric intuition, while the positive characteristic case is even more algebraic and abstract. You also design and produce film visualizations of algebraic surfaces that fascinate the viewer with their vivid and spontaneous graphical quality. How do you explain this apparent contradiction?

That’s right, seemingly contradictory interests are at play here. But they also have things in common. When I encountered resolution of singularities in the nineties, skepticism kept me from thoroughly entering the subject for a long time. I got hooked when I tried to understand in my own language the more recent proofs by Villamayor and Bierstone-Milman of Hironaka’s original result, and to lift them to a more conceptual level. It seemed as if a beautiful mansion had opened its door. Since the characteristic zero assumption was spread throughout the proofs of that time, I also wanted to localize more precisely the special problems that occur in positive characteristic. Although these proofs are, of course, pure algebra, their logical structure of mutual induction is set in the context of a huge, interwoven architecture. To a certain extent this structure is also geometry, but a geometry of proof, of reasoning, like the structure of a symphony, or the construction of a cathedral, or the mechanism of a clock. It is the geometry of analytical thinking.

And sometime around then the visualizations started ...

Yes, I still remember the moment, somewhere around 2002, when Josef Schicho turned up with the POV-Ray program. But in fact, my involvement with geometric figures started much earlier, shortly after my PhD, when I crafted paper models of singular surfaces for my talks and classes. This was not a very straightforward thing to do, since paper cannot be stretched to make bows. I had to insert small slits to allow for the curvature. These paper models still exist, in a Christmas cookie tin.

New opportunities arose because of POV-Ray.[2] The program is intended for the production of scenes for the video and advertising industry, creating virtual landscapes and spaces.

However, the POV-Ray is also able to represent graphically the zero-sets of polynomial equations in three variables—real algebraic surfaces. This is done with amazing accuracy and beauty.

Couldn't this also be done using available computer algebra programs like Maple or Mathematica?

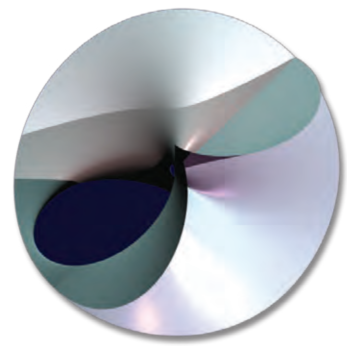
Not with comparable mathematic or aesthetic quality. Even though POV-Ray does occasionally display artifacts, such as fringes and wrinkles, that are inconsistent with the mathematical reality, the results are in general very satisfactory.

And then you started to generate the classical surfaces on the computer ...

Not only the classical surfaces; there was also a great deal of trial and error that resulted in new surfaces. Soon it became clear that there are certain criteria and demands that need to be considered: simplicity, plainness, naturality, vividness, mathematical relevance, modesty.

And how does this work out in practice? What do you have to do when "inventing" a new surface?

Suppose we start with the equation $x^2yz + xy^2 + y^3 + y^3z = x^2z^2$, without first contemplating its geometric configuration. An algebraic geometer who is only equipped with pencil and paper will obviously first determine some basic data: the irreducible components, the singular locus, the tangent cone, the intersection curves with planes and spheres. With POV-ray you simply turn on the machine, type the equation and press the "enter" key. In a few seconds (with such simple equations), a picture is produced



Solitude, of equation $x^2yz + xy^2 + y^3 + y^3z = x^2z$

And what a picture! Most often you don't see anything, or only shadows and splotches of color. To arrive at a reasonable result, you have to know how POV-Ray works. There are also the more recent programs such as Surf, Surf-ex, and Surfer.[3]

Which you will now explain to us!

Yes, the principle is really simple, and precisely what you would devise if you had to develop the program yourself. (The real challenge lies in the technical aspects of programming, such as optimization of the running time.)

Imagine that you have a spatial object (for instance, the solution set of an equation) which you want to render as accurately as possible. The naive method would be to compute all solutions of the equation in a sufficiently fine grid. This would give you a collection of 3D-data that you could represent visually, say by making a photograph.[4] If you only want a photo (i.e., 2D-data), the effort can be substantially reduced.

We place—virtually—a camera in space by fixing its position, viewing direction, and angle of aperture. This determines a real cone, which we fill with rays emanating from the camera, their density adjusted to the desired degree of precision. We intersect each of these rays with the object, which in the case of algebraic surfaces corresponds to solving a polynomial equation in one variable. This works rather quickly and, for one ray, often produces several solutions (since there can be several intersection points). The computation is more delicate when a ray hits the surface tangentially; in that case, artifacts may appear. Once all intersection points on the various rays are calculated, it is routine to compose the complete picture from them.

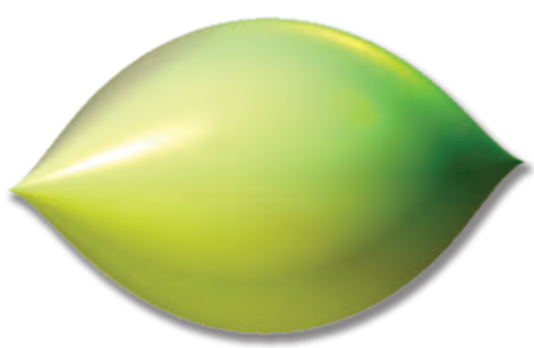
I Spy With My Little Eye by Herwig Hauser continued

That seems pretty clear, but is it really as simple as that?

The difficulty lies in the choice of parameters. If we don't know what the surface looks like (which we have to assume to be the case), we must first try to get a rough impression, say by looking from longer distances and from different angles.[5] Then the real work begins: the choice of perspective, position, intensity and color of the light source, degree of reflection (respectively, transparency) of the surface material, and the coefficients in the equation. Here POV-Ray is unsurpassed, and with some practice you can achieve very nice results. It really takes on the roles of the lighting technician and the stage designer in theater. The stage designer has to compose the scene with its moving actors and static scenery. And some surfaces are as stubborn as spoiled actors, who play hard to get! The tricky geometry of a singularity often prevents us from clearly representing it in a convincing manner.

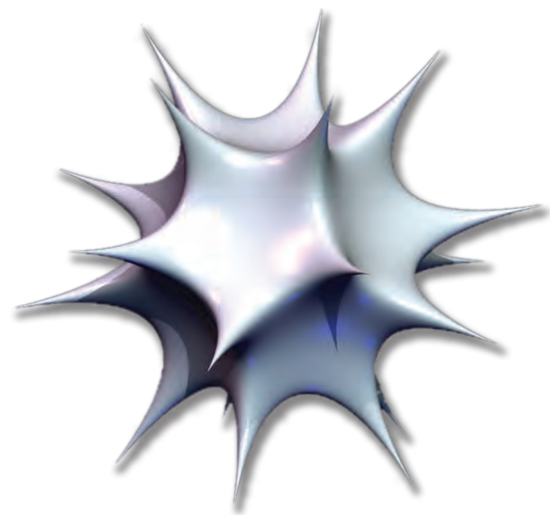
As you suggested before, the choice of equation doesn't need to be the only approach to visualization, does it?

No, we can also go in the other direction, an equation for a surface with given properties. A simple example is the surface Zitrus, for which we can find a global equation from our knowledge of local equations at the two singular points.



Zitrus, of equation $x^2+z^2 = y^3(1-y)^3$

Even though it is very simple, many people find Zitrus quite appealing. For Platonic Stars we had to use some basic invariant theory to find with a suitable equation.



The Dodecahedral Star, one of the Platonic Stars (with complicated equation)

In this case, the choice of parameter values was very subtle, since these values strongly influence the aesthetics of the resulting form.

Daisy, of equation $(x^2-y^3)^2 = (z^2-y^2)^3$

The surface Daisy was also developed by first imagining its shape. The goal was to find a surface that is singular along an entire curve, which itself should be singular. Furthermore, the transversal sections to this curve should again be singular, namely cusps.

Are there new mathematical problems that have emerged from your work on visualization?

Yes, several. First, there is the problem of finding an equation for a surface with a given geometric configuration.[6] A deeper problem is to codify geometric shapes using a mathematical language that is not based on algebraic expressions such as implicit equations. While such equations are the most compact way of describing a surface, it can be difficult to extract the geometric information from them. It is clear that our brain—when seeing a surface—stores the data in a way that allows us to reproduce the image from memory (these data are most probably not the coefficients of an equation). Up to now there is no means to communicate this type of data: look at Daisy for two minutes, then go to your colleague next door and try to describe the geometry so that she is able to draw an accurate picture of the surface. The development of such a language could be a new goal for Algebraic Geometry; it would go far beyond the traditional, algebraic-logical based mathematics. And finally, the visualization feeds back on the resolution of singularities. Such a resolution is, roughly speaking, a parametrization of the singular surface by a smooth surface. If we could compute resolutions efficiently (it seems that we are still far away from doing this), we could represent surfaces exactly and without errors. Using quick resolution would be the Rolls Royce of visualization.

To conclude, a few words about your movie "ZEROSET – I spy with my little eye." A prominent mathematician said, after viewing the film at its premiere in Madrid: "Great work —but rather wierd." Do you know what he meant by this?

This premiere was quite impressive—a full auditorium, with mathematicians taking pictures and videos during the presentation. The success was overwhelming and very astonishing. For the movie does not explain anything (in contrast to what mathematics aims to do), and it contains only a small number of mathematical pictures. Between the pictures are a series of snapshots of real scenes—cartoons, formulas, the writing of a love story—everything built on the main element of the film, the music. Most of this is the work of my student, Sebastian Gann, who spent many, many hours producing the POV-Ray animations, after which he composed the various parts for the video. The credit goes to him and his team.

It was important for us not to focus on a single subject. The movie is a modest attempt to see and represent mathematics in a larger cultural context as something beautiful and exciting.

Your dream, your vision?

To build a truly huge sculpture of a singular algebraic surface, as for example Anish Kapoor does in the smooth (non-algebraic) case with his Chicago Cloud Gate. Or, still better, to produce the surface Solitude on a large scale as the stage design for the Bregenz Festival.[7]

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> *"[My dream vision is] to build a truly huge sculpture of a singular algebraic surface...Or, to produce the surface Solitude on a large scale...*

1 The pictures and online video clips were produced by Alexandra Fritz from the University of Vienna. The interviewer is indebted to Karl Knight for very valuable support in polishing the translation.

2 www.povray.org.

3 See www.imaginary2008.de/surfer.php.

4 In many situations it is advantageous to obtain 3D-data, for instance for 3D-prints or model-building.

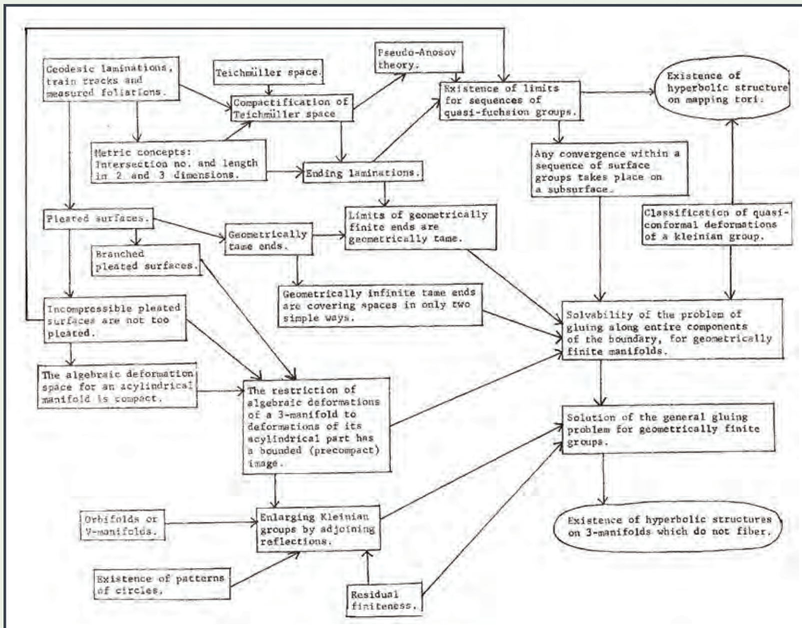
5 Various programs permit one to move the camera in real time, or, said differently, to make the surface rotate.

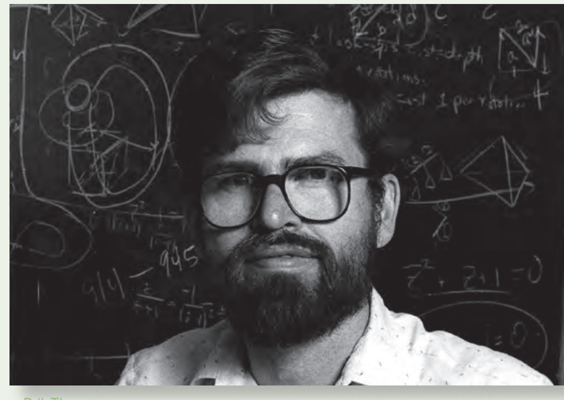
6 This has led to a current challenge where people try to visualize everyday objects such as cherries, coffee cups, and snowmen using algebraic surfaces.

7 www.bregenzerfestspiele.com.

Thurston's Geometrization Conjecture

by David Gabai and Steven Kerckhoff





Bill Thurston: Photo by Michael Pirrocco, courtesy of the American Mathematical Society

Steven Kerckhoff

David Gabai.
Photo courtesy of Reina Gabai

Introduction

In the late 1970s Bill Thurston proved his hyperbolization theorem for Haken manifolds and stated his geometrization conjecture for closed 3-manifolds. With Perelman's spectacular proof it is appropriate to review the history of this conjecture and be reminded of the enormous influence of Thurston's work towards geometrization on many other subfields of mathematics.

Two dimensional manifolds

By the end of the nineteenth century all the closed connected 2-manifolds were discovered and it was known that each supported a metric of constant curvature. (Unless otherwise stated, all manifolds in this article are orientable.)[1]

Theorem 2.1. *If S is a closed connected orientable surface, then S is homeomorphic to either the sphere, torus or a surface of genus g for some $g \geq 2$. If S is a sphere, then S has a metric of constant curvature $+1$, if S is a torus, then it has a metric of constant curvature 0 and if S is a surface of genus g, $g \geq 2$, then S has a metric of constant curvature -1.*

[1] A rigorous classification of triangulable surfaces was not achieved until 1907 by Dehn and Heegaard. In 1925, Rado completed the classification theorem by showing that any topological surface is triangulable.

The surfaces that support a metric of curvature $+1$ (i.e., the sphere) are exactly those that have a finite fundamental group (i.e., one element in the case of the sphere). The surfaces S that support a metric of curvature zero (i.e., the torus) are exactly those with $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(S)$. ($\pi_1$(torus)$= \mathbb{Z} \oplus \mathbb{Z}$) . The surfaces that support a metric of constant curvature -1 are exactly those with infinite fundamental group that do not have $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup. Amazingly, a similar but immensely deeper result holds for 3-dimensional manifolds.

Thurston's geometrization conjecture

We say that a 3-manifold M is *prime* if every smoothly embedded separating 2-sphere bounds a 3-ball. If M is not prime, then we obtain two 3-manifolds by splitting M along an essential separating 2-sphere and capping off each 2-sphere boundary component with a 3-ball. The opposite operation is called *connected sum*. It is a theorem of Kneser that any closed orientable 3-manifold can be finitely decomposed in this manner into a union of prime 3-manifolds. By a result of Milnor, the set of 3-manifolds so obtained is uniquely determined. There is the closely related concept of a 3-manifold M being *irreducible*. This means that every smoothly embedded 2-sphere in M bounds a 3-ball. The only closed 3-manifold that is both reducible and prime is $S^2 \times S^1$.

Thurston's geometrization conjecture 3.1. *Let M be a closed irreducible 3-manifold. Then exactly one of the following holds:*

i) $\pi_1(M)$ is finite and M has a metric of constant curvature $+1$.

ii) $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$.

iii) M has a metric of constant curvature -1.

Thurston's original formulation involved elaborating possibility ii), although by 1990 the geometrization conjecture was reduced to this statement. Thurston's original formulation states that an irreducible 3-manifold, after possibly cutting along a canonical collection of disjoint tori[2], splits into pieces each of which supports one of eight homogeneous geometries.[3]

An astounding feature of Thurston's conjecture was that it subsumed many very interesting long-standing conjectures, of which the Poincaré conjecture was the most famous. For example, there was the *spherical space form problem* that a finite group that acts smoothly and freely on the 3-sphere is isomorphic to a subgroup of $SO(4)$ and there was the *linearization conjecture* that such actions are conjugate to actions by isometries. Both are implied by case i). There was also the *Seifert fibered space conjecture*, proved in 1990[4], a consequence of which reduced Thurston's conjecture to the statement 3.1 above. There was also the *universal covering* $\mathbb{R}^3$ *conjecture* for closed aspherical 3-manifolds and the *residual finiteness of 3-manifold groups* conjecture.

[2] This is the Jaco-Shalen, Johannson (JSJ) characteristic manifold.

[3] In the hyperbolic case the metric is complete of finite volume and is defined on the interior of the piece.

[4] This was the culmination of the work of many mathematicians over a thirty year period.

Thurston's Geometrization Conjecture

by David Gabai and Steven Kerckhoff continued

Historical Remarks

Although the above description of the geometrization conjecture suggests that it is a natural outgrowth of the geometric classification of surfaces, this is far from the case. Indeed, before Thurston's work in this area, there was virtually no evidence for such a conjecture. The idea that one could use homogeneous geometry to study the topology of 3-manifolds was truly revolutionary. In large part, this was because there were so few examples known of closed hyperbolic 3-manifolds. Surfaces of a fixed genus have many different hyperbolic structures, and it is easy to construct examples. In contrast, a hyperbolic structure on a closed 3-manifold is, by Mostow rigidity, unique up to isometry. This suggests that construction of such a structure should be an extremely delicate matter. In 1912, Gieseking found the first example of a complete, finite volume hyperbolic 3-manifold (which was nonorientable). The first closed examples were found in 1931 by Lobell and in 1933 by Seifert and Weber. The collection of known examples was tiny.

Much of the field of Kleinian groups, (discrete groups of $PSL(2, C)$), developed out of parts of complex analysis. The work of Ahlfors, Bers, and others on quasi-conformal maps provided a robust theory of geometrically finite Kleinian groups, particularly those with a non-trivial domain of discontinuity under the action by linear fractional transformations on the 2-sphere. One of the main results of the Ahlfors-Bers theory is that these groups are parametrized by conformal structures on the Riemann surfaces obtained as quotients of the domains of discontinuity. These groups also act on hyperbolic space with quotient a hyperbolic 3-manifold with infinite volume (so Mostow rigidity does not apply). The theory of 3-manifolds rarely played a role in this field before 1970.

In a seminal paper published in 1972, Al Marden found important connections between Kleinian groups and 3-manifold theory. Among other things he showed how the the Klein-Maskit combination theorem corresponds to Haken's decomposition along incompressible surfaces. At the end of that paper he offered "... two problems which appear insurmountable at this time." The first became known as the Marden tameness conjecture[5]. The second asked for necessary and sufficient conditions on the fundamental group of a 3-manifold for that manifold to have a hyperbolic structure.

Also in the early 1970s new but very specialized examples of finite volume hyperbolic manifolds were discovered. In 1975, Bob Riley showed that the figure-eight knot complement had a complete, finite volume hyperbolic structure. He also stated the necessary condition for a knot complement to have a hyperbolic structure, namely, that all of the $\mathbb{Z} \oplus \mathbb{Z}$ subgroups of π_1 be peripheral.[6] Around the same time, Troels Jørgensen constructed finite volume hyperbolic structures on many punctured torus bundles over S^1. Thurston was strongly influenced by these examples.

[5] A Clay Research Award was just given for its resolution.

[6] This is the finite volume analogue of condition ii) not occurring.

Initially, Thurston was dubious that a 3-manifold fibering over the circle (such as the figure-eight knot complement) could be hyperbolic, for such a structure would probably have the following striking consequence. The fiber would lift to an open disk in hyperbolic 3-space whose ideal boundary would be a space-filling curve in the 2-sphere at infinity.[7] He then began his analysis of surface diffeomorphisms that led to the theory of pseudo-Anosov diffeomorphisms and his compactification of Teichmüller space. The geometry of these maps as well as the Riley and Jorgensen examples convinced him to reverse his initial opinion, and he began to believe that the general hyperbolization theorem for 3-manifolds[8] might be true.

Major breakthroughs occurred at a frenzied pace during the 1976-1977 academic year. Thurston began his graduate course at Princeton by discussing the JSJ decomposition of 3-manifolds and conjecturing that, when the decomposition is non-trivial, the non-Seifert fibered pieces have finite volume hyperbolic structures. Later, he discussed the Ahlfors-Bers theory and explained how a gluing problem for conformal structures (later called the "skinning map") provided an inductive approach to the conjecture. At the beginning of the spring semester, he announced that he had solved the gluing problem and could solve the hyperbolization conjecture in the case of an irreducible, sufficiently large 3-manifold (called a *Haken manifold*), except when it fibers over S^1. By the end of the semester, the fibering case had also fallen.

During the next two academic years, Thurston's courses at Princeton revolved around his proof of his Haken theorem. However, it quickly became clear that he viewed it as a special case of a general geometric theory of 3-manifolds. It is unclear exactly when the geometrization conjecture was first stated, but by 1978 it was being discussed in the form we know it today.

After Thurston's Haken geometrization theorem, there were two main developments that seemed to lead toward the general geometrization conjecture. One was the solution of the Seifert-fibered space conjecture, which reduced the conjecture to the spherical and hyperbolic cases (cases i) and iii) above). The other was the orbifold theorem which proved the geometrization conjecture for 3-dimensional orbifolds (definition given below) with non-trivial 1-dimensional local fixed point set. Announced by Thurston in 1982, it was the first general theorem that dealt with the spherical case as well as the other geometries. Part of the proof utilized Hamilton's pioneering work on 3-manifolds with positive Ricci curvature, a harbinger of the role that Ricci flow was later to play. However, it would be another twenty years before Perelman would bring this to fruition.

[7] This was ultimately proven in a beautiful paper of Jim Cannon and Thurston.

[8] That is, if a manifold does not satisfy i) and ii) above, it is hyperbolic.

Thurston's Geometrization Conjecture

by David Gabai and Steven Kerckhoff continued

Remarks on the Proof of Thurston's Hyperbolization theorem

In the late 1970s Thurston's result was known as the *monster theorem*. It was an incredible amalgam of immensely original work from a wide range of (at the time) disparate mathematics. Various pieces of the argument (or its byproducts) invigorated or created many subfields of mathematics that are distinct from geometrization. Many of these are visible in Thurston's flow chart created for a conference at Bowdoin College in the summer of 1980; they include those that I describe below.

Automorphisms of surfaces

Thurston found a model for a generic surface diffeomorphism, called a pseudo-Anosov diffeomorphism, which generalizes the stretch-squeeze map of the torus induced by a linear diffeomorphism with two distinct real eigenvalues. In the process, he rediscovered important theorems of Nielsen and created objects of study that have tremendously influenced geometric group theory, ergodic theory, and 3-dimensional topology.

Measured foliations of surfaces

The boundary of Thurston's compactification of Teichmüller space is a space of measured foliations that is a completion of the set of simple closed curves on a surface. Remarkably, these foliations are the topological incarnation of those determined by holomorphic quadratic differentials, a central object in Riemann surface theory. This put in a new context various density conjectures in that theory and they remain a central tool in Teichmüller theory, polygonal billiards, interval exchange maps, and numerous parts of low-dimensional topology.

Geometrically finite groups and pleated surfaces

Thurston provided a tight internal geometric structure on the ends of the hyperbolic 3-manifold determined by a geometrically finite Kleinian group by filling them with families of pleated surfaces. The surfaces are isometric to hyperbolic surfaces but are bent along a union of geodesics, called a geodesic lamination. These laminations are the hyperbolic geometric realization of a measured foliation and provide strong compactness properties for these groups.

Classification of Kleinian groups and ending laminations

Thurston showed that certain limits of geometrically finite groups possess a structure at infinity called an ending lamination which is a geodesic lamination that can be viewed as a limit of the laminations from pleated surfaces exiting an end. He conjectured that finitely generated Kleinian groups should be classified by a combination of the conformal structures at infinity (as in the Ahlfors-Bers theory) and the ending laminations. Called the *ending lamination conjecture*, it became a central conjecture in the field and provided the basis for a tremendous amount of research. It has been recently solved by Brock, Canary, and Minsky.

Extension problem and local connectivity

The lift of an incompressible surface in a hyperbolic 3-manifold to its universal cover is a topological disk that can have a very complicated behavior at infinity. Cannon and Thurston showed that its ideal boundary maps to a space-filling curve when the surface is the fiber of a fibering over S^1. This raised the general question of when maps of certain disks extend to the boundary. Closely related is the question of whether or not the limit set of a general finitely generated Kleinian group is locally connected, a problem that pre-dates Thurston's work. It has recently been solved by Mj.

Andreev's theorem and circle packings

In the Haken manifold proof, a hyperbolic structure is built up by gluing together simpler pieces that inductively possess hyperbolic structures. The initial step, when the pieces are balls, is solved by putting the structure of a polyhedron on the boundary. In the process, Thurston gave a new proof of Andreev's existence and uniqueness theorem for convex hyperbolic polyhedra, and then greatly generalized it. The structures were also interpreted as circle packings, generating an entirely new theory of these as well.

Theory of orbifolds

An orbifold is locally modeled on the quotient of an open set in Euclidean space by a finite group in the way a manifold is locally modeled on a Euclidean open set. Thurston showed that this concept, which was introduced by Satake under the name of V-manifold, is a flexible and powerful tool when applied to geometric structures. Orbifolds arise in his proof when interpreting the polyhedra described above as groups generated by reflections in the faces; the local fixed-point set consists of the entire boundary that has been "mirrored." Examples of hyperbolic manifolds can be constructed by taking finite index subgroups of the reflection groups.

Hyperbolic Dehn filling

Thurston's hyperbolization theorem provides complete finite volume hyperbolic structures on the interior of many compact manifolds with torus boundary, including most knot complements[9]. Although the complete structure is unique, Thurston showed that there are many nearby incomplete structures that, when completed, give a smooth hyperbolic structure on a closed manifold that is obtained by attaching a solid torus to the boundary. There are an infinite number of topologically distinct ways to attach the solid torus, and all but a finite number can be given a hyperbolic structure in this way. Thus, a single non-compact hyperbolic structure gives rise to an infinite number of closed ones. This process, called hyperbolic Dehn filling, is a purely 3-dimensional phenomenon and helps explain why hyperbolic manifolds are so prevalent in dimension 3 but not in higher dimensions. The study of hyperbolic Dehn surgery quickly became a central topic in the field.

[9] Indeed, all knot complements and all compact manifolds whose boundary is a nonempty union of tori are covered by this theorem, i.e., it determines exactly which of these have a hyperbolic structure on its interior.

Thurston's Geometrization Conjecture

by David Gabai and Steven Kerckhoff continued

Non-Haken manifolds

In analyzing hyperbolic Dehn surgery on a single example, the figure-eight knot complement, Thurston showed that all but a finite number of the topological manifolds obtained from it by Dehn filling are non-Haken. This topological result was the first of its kind, spinning off an entire research area. It also provided a huge number of hyperbolic manifolds that were not covered by the original Haken geometrization theorem.

Volumes of hyperbolic 3-manifolds

Based on a few hand and computer calculations, Thurston conjectured that there are interesting connections between the volume of a hyperbolic 3-manifold and its topological complexity. He used work of Jørgensen and Gromov to show that the set of volumes is a closed well-ordered set of type ω^ω. This created the very active new subfield of (among many other things) finding the lowest volume manifolds of various topological types and finding various topological constraints on low-volume manifolds.

Conclusion

Bill Thurston made a bold and revolutionary conjecture. The mathematics he discovered trying to prove it was (and is) tremendously influential, extending far beyond that of geometrization itself.

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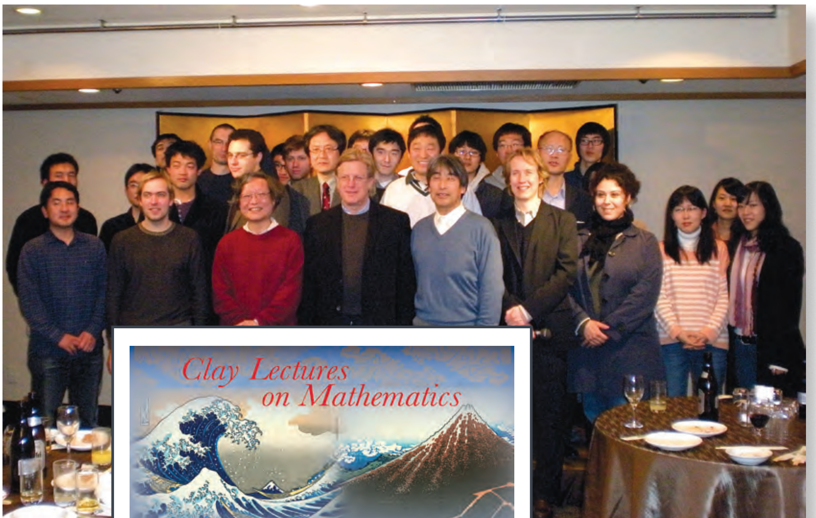
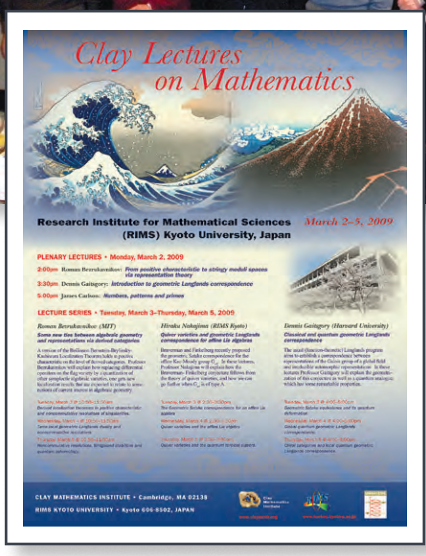
Clay Lectures on Mathematics

Lectures at RIMS, Kyoto University





Organizers

David Ellwood (CMI)
Masaki Kashiwara (RIMS)
Hisashi Okamoto (RIMS)

The 2008/2009 Clay Lectures were organized in collaboration with the Research Institute for Mathematical Sciences (RIMS) in Kyoto, Japan, which graciously hosted the event.

The lecturers were Roman Bezrukavnikov (MIT), Dennis Gaitsgory (Harvard), and Hiraku Nakajima (RIMS). The event included a public lecture by CMI President Jim Carlson, entitled *Numbers, Patterns, and Primes*.

Each Clay Lecturer delivered a series of three talks aimed at introducing the audience to research in his specialty. Roman Bezrukavnikov and Dennis Gaitsgory also gave public lectures entitled *From Positive Characteristic to Stringy Moduli Spaces via Representation Theory* and *Introduction to Geometric Langlands Correspondence*, respectively. These public talks aimed to motivate graduate students and researchers in other fields.

Roman Bezrukavnikov spoke on *Some New Ties between Algebraic Geometry and Representations via Derived Categories*, beginning with a version of Beilinson-Bernstein-Brylinsky-Kashiwara localization theorem which holds in positive characteristic on the level of derived categories. Hiraku Nakajima spoke on *Quiver Varieties and Geometric Langlands Correspondence for Affine Lie Algebras*, beginning with an exploration of the geometric Satake correspondence for the affine Kac-Moody group G_{aff} recently proposed by Braverman-Finkelberg. Dennis Gaitsgory spoke on *Classical and Quantum Geometric Langlands Correspondence*, begining with the geometrization suggested by Drinfeld in which the space of automorphic functions is replaced by the category of automorphic sheaves.

Clay Lectures on Mathematics

Clay-Mahler Lectures, Australia

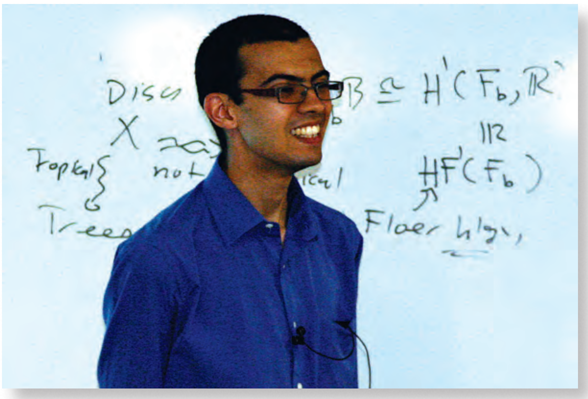
July 23 - October 7, 2009 Melbourne - Perth - Brisbane - Sydney - Canberra - Adelaide

Organizers

Alan Carey (ANU)
David Ellwood (CMI)
Andrew Hassell (ANU)

The 2009/2010 Clay Lectures was organized in collaboration with the Australian Mathematical Society and the Australian Mathematical Sciences Institute to create a national event that took place in six cities and fourteen institutions. The lecturers were Terry Tao (Fields Medalist, Clay Research Awardee, and former Clay Research Fellow), Danny Calegari (Clay Research Awardee and former Clay LiftOff Fellow), and Mohammed Abouzaid (current Clay Research Fellow).

The outstanding success of Australian mathematicians in recent years has been highlighted by the prominence of Australian recipients of Clay awards. The Clay-Mahler tour was organized to celebrate that success through a series of thirty-six talks at many of Australia's leading educational establishments. All three Clay Lecturers delivered plenary lectures at the annual meeting of the Australian Mathematical Society at the University of South Australia in Adelaide. The Clay-Mahler Lecture tour was comprised of a series of talks at different levels and accessible to audiences ranging from members of the general public (public lectures) to students studying mathematics at the undergraduate or post graduate level (colloquium talks) to professional mathematicians interested in the state of the art of a particular field (specialist talks). Listed below are abstracts for the various talks, together with a calendar of the event.



Mohammed Abouziad (MIT / CMI)

Mohammed Abouzaid (MIT & CMI)

Colloquium talks

Understanding hypersurfaces through tropical geometry

Given a polynomial in two or more variables, one may study the zero locus from the point of view of various mathematical subjects (number theory, algebraic geometry, etc.). In his talk Abouzaid explained how tropical geometry allows one to encode topological quantities by combinatorial objects now known as tropical varieties.

Functoriality in homological mirror symmetry

Kontsevich's original version of the homological mirror symmetry conjecture was a statement about pairs of Calabi-Yau manifolds, with no indication of any connection between mirrors of varieties that are related to each other. In this talk Abouzaid described recent progress which reveals situations in that homological mirror symmetry exhibits more "functorial" properties. This conjectural functoriality is clearest for the case of the inclusion of an anticanonical divisor in a Fano variety. The talk focused on examples starting in dimension one, and on explaining the geometric source of these phenomena.

Specialist talks

A mirror construction for hypersurfaces in toric varieties (Parts I & II)

The Strominger-Yau-Zaslow conjecture gives an intrinsic explanation for homological mirror symmetry in the case of Calabi Yau manifolds. In this talk Abouzaid explained that by extending the SYZ conjecture beyond the Calabi-Yau case, one may associate a Landau-Ginzburg mirror to generic hypersurfaces in toric varieties. The key idea is to use tropical geometry to reduce the problem to understanding the mirror of hyperplanes.

String topology and the Fukaya category of cotangents bundles

The most interesting version of the Fukaya category of cotangent bundles includes Lagrangians that are allowed to be non-compact. Abouzaid explained how this category is equivalent to the category of modules over the based loop space. The "classical" equivalence between symplectic homology and the homology of the based loop space (with the pair of pants product on one side and the Chas-Sullivan product on the other) follows from this story.

A topological model for the Fukaya category of plumbings

The simplest examples of symplectic manifolds beyond cotangent bundles are obtained by plumbing. In his talk Abouzaid explained a topological model for the Fukaya categories of these manifolds; the model is given in terms of classical invariants from algebraic topology (the cochain complexes on the skeleta).

Terry Tao (UCLA)

Terry Tao (UCLA)

Public lectures

Structure and randomness in the prime numbers

"God may not play dice with the universe, but something strange is going on with the prime numbers." — Paul Erdös. The prime numbers are a fascinating blend of both structure (for instance, almost all primes are odd) and randomness. It is widely believed that beyond the "obvious" structures in the primes, the primes otherwise behave as if they were distributed randomly; this "pseudorandomness" then underlies our belief in many unsolved conjectures about the primes, from the twin prime conjecture to the Riemann hypothesis. Although this pseudorandomness has been frustratingly elusive to prove rigorously, there has been recent progress in capturing enough of this pseudorandomness

Clay Lectures on Mathematics

Clay-Mahler Lectures, Australia continued

to establish new results about the primes. Among these new results is the fact that they contain arbitrarily long arithmetic progressions. Tao surveyed some of these developments in his talk.

The cosmic distance ladder

How do we know the distances from the earth to the sun and moon, from the sun to the other planets, and from the sun to other stars and distant galaxies? Clearly, we cannot measure these directly. Nevertheless, there are many indirect methods of measurement, combined with basic high-school mathematics, which allow one to get quite convincing and accurate results without the need for advanced technology. For instance, even the ancient Greeks could compute the distances from the earth to the sun and moon with moderate accuracy. These methods rely on climbing a "cosmic distance ladder," using measurements of nearby distances to then deduce estimates on distances slightly further away. In this talk Tao discussed several rungs in this cosmic distance ladder.

Mathematical research and the Internet

Prof. Tao discussed some personal experiences of how the Internet is transforming the way he, and other mathematicians, do research. These range from such mundane tools as email, home pages, and search engines, to blogs, preprint servers, wikis, and more.

Colloquium talks

Recent progress in additive prime number theory

Additive prime number theory is the study of additive patterns in the primes. Tao surveyed some recent advances in this subject, including the results of Goldston, Pintz, and Yildirim on small gaps between primes, the results of Green and himself on arithmetic progressions in the primes, and the results of Bourgain, Gamburd, and Sarnak for detecting almost primes in orbits.

Compressed sensing

Suppose one wants to recover an unknown signal x in R^n from a given vector Ax=b in R^m of linear measurements of the signal x. If the number of measurements m is less than the degrees of freedom n of the signal, then the problem is underdetermined and the solution x is not unique. However, if we also know that x is sparse or compressible with respect to some basis, then it is a remarkable fact that (given some assumptions on the measurement matrix A) we can reconstruct x from the measurements b with high accuracy, and, in some cases, with perfect accuracy. Furthermore, the algorithm for performing the reconstruction is computationally feasible. This observation underlies the newly developing field of compressed sensing. In this talk Tao discussed some of the mathematical foundations of this field.

The proof of the Poincaré conjecture

In a series of three terse papers in 2003 and 2004, Grisha Perelman made spectacular advances in the theory of the Ricci flow on 3-manifolds, leading in particular to his celebrated proof of the Poincaré conjecture (and most of the proof of the more general geometrization conjecture). Remarkably, while the Poincaré conjecture is a purely topological statement, the proof is almost entirely analytic in nature, in particular relying on nonlinear PDE tools together with estimates from Riemannian geometry to establish the result. In this talk Prof. Tao discussed some of the ingredients used in the proof, and sketched a high-level outline of the argument.

Specialist talks

Discrete random matrices

The spectral theory of continuous random matrix models (e.g., real or complex gaussian random matrices) has been well studied, and very precise information on the distribution of eigenvalues and singular values is now known. However, many of the results rely quite heavily on the special algebraic properties of the matrix ensemble (e.g., the invariance properties with respect to the orthogonal or unitary group). As such, the results do not easily extend to discrete random matrix models, such as the Bernoulli model of matrices with random ±1 signs as entries. Recently, however, tools from additive combinatorics and elementary linear algebra have been applied to establish several results for such discrete ensembles, such as the circular law for the distribution of eigenvalues, and also explicit asymptotic distributions for the least singular values of such matrices. Tao surveyed some of these developments in his talk.

Arithmetic progressions in the primes

A famous and difficult theorem of Szemerédi asserts that every subset of the integers of positive density will contain arbitrarily long arithmetic progressions; this theorem has had four different proofs (graph-theoretic, ergodic, Fourier analytic,

Program Overview

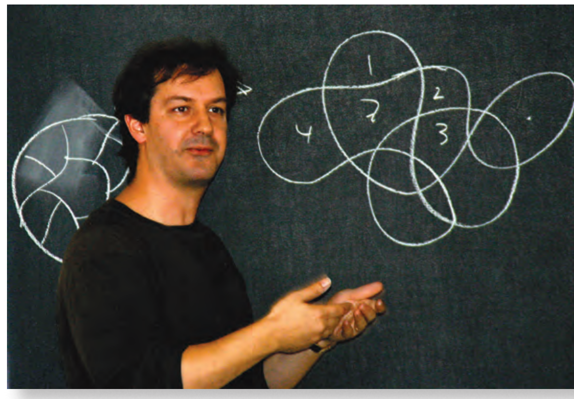
and hypergraph-theoretic), each of which has been enormously influential, important, and deep. It had been conjectured for some time that the same result held for the primes (which of course have zero density). Tao discussed recent work with Ben Green establishing this conjecture, by viewing the primes as a subset of the almost primes (numbers with few prime factors) of positive relative density. The point is that the almost primes are much easier to control than the primes themselves, thanks to sieve theory techniques such as the recent work of Goldston and Yildirim. To "transfer" Szemerédi's theorem to this relative setting requires that one borrow techniques from all four known proofs of Szemerédi's theorem, and especially from the ergodic theory proof.

Wave maps

The wave map equation is one of the fundamental geometric wave equations, being on the one hand the dynamic analogue of harmonic maps, and a simplified model for the Einstein equations and gauge field theories, such as the Yang-Mills equations, on the other. In recent years there has been substantial progress in understanding basic questions such as global regularity and singularity formation for this equation using new tools such as the induction-on-energy strategy of Bourgain, the concentration-compactness technology of Kenig and Merle, a geometric gauge fixing arising from the harmonic map heat flow, and even some limiting arguments used by Perelman in his proof of the Poincaré conjecture. In his talk Tao surveyed some of these developments.

Recent progress on the Kakeya problem

The Kakeya needle problem asks: is it possible to rotate a unit needle in the plane using an arbitrarily small amount of area? The answer is known to be yes, but analogous problems in higher dimensions (where one now seeks to find sets of small dimension that contain line segments in all directions) remain open, and are related to many other important conjectures in harmonic analysis, PDE, and even number theory and computer science. There have been many partial results on this problem, using such diverse techniques as geometric measure theory, incidence combinatorics, additive combinatorics, and PDE; more recently, algebraic geometry, and even algebraic topology have been used to obtain new breakthroughs in this subject. Tao discussed many of these new developments in his talk.



Danny Calegari (Caltech)

Danny Calegari (Caltech)

Colloquium talks

Faces of the stable commutator length norm ball

It often happens that a solution of an extremal problem in geometry has more regularity and nicer features than one has an a priori right to expect. In his talk, Calegari explained how a simple topological problem—when does an immersed curve on a surface bound an immersed subsurface?—is unexpectedly related to linear programming in nonseparable Banach spaces, and gives rise to geometric and dynamical rigidity and discreteness of symplectic representations.

Specialist talks

Stable commutator length (scl) answers the question of: "what is the simplest surface in a given space with prescribed boundary?" where "simplest" is interpreted in topological terms. This topological definition is complemented by several equivalent definitions—in group theory, as a measure of non-commutativity of a group; and in linear programming, as the solution of a certain linear optimization problem. On the topological side, scl is concerned with questions such as computing the genus of a knot, or finding the simplest 4-manifold that bounds a given 3-manifold. On the linear programming side, scl is measured in terms of certain functions called quasimorphisms, which arise from hyperbolic geometry (negative curvature) and symplectic

Clay Lectures on Mathematics

Clay-Mahler Lectures, Australia continued

geometry (causal structures). In his lectures, Calegari discussed how scl in free and surface groups is connected to such diverse phenomena as the existence of closed surface subgroups in graphs of groups, rigidity and discreteness of symplectic representations, bounding immersed curves on a surface by immersed subsurfaces, and the theory of multidimensional continued fractions and Klein polyhedra.

Surface subgroups from homology

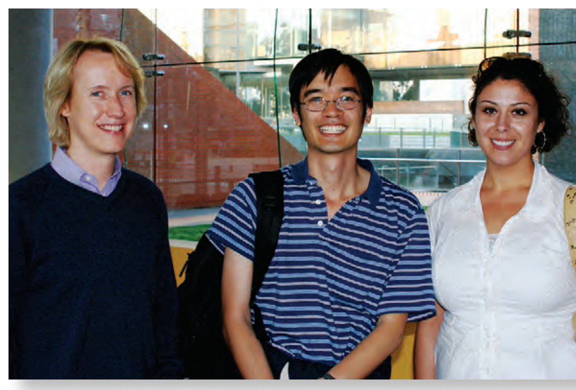
A two-sided embedded surface in a 3-manifold is either injective in π_1, or can be simplified by a "compression," using Dehn's lemma (famously proved by Papakyriakopoulos). It follows that embedded surfaces of least genus representing an integral homology class are injective in π_1, and therefore the fundamental groups of many 3-manifolds contain surface subgroups. For more complicated groups or spaces, no tool resembling Dehn's lemma exists; nevertheless, if G is a graph of free groups amalgamated along cyclic subgroups, we show that every rational class in $H_2(G;Q)$ is (virtually) represented by a map of a surface group of least Gromov norm, and such a map is injective. In particular, such groups often contain surface subgroups.

Faces of the scl norm ball

An immersed loop in the plane might or might not bound an immersed disk, and if it does, the disk it bounds might not be unique. An immersed loop on a surface might not bound an immersed subsurface, but admit a finite cover which does. Most homologically trivial geodesics on hyperbolic surfaces with boundary do not even virtually bound an immersed surface. However, we show that *every* homologically trivial geodesic in a *closed* hyperbolic surface virtually bounds an immersed surface, and every homologically trivial geodesic in a hyperbolic surface with boundary virtually cobounds an immersed surface together with a sufficiently large multiple of the boundary. This gives rise to a codimension one face in the unit ball in the scl norm of a free group associated to each realization of the free group as the fundamental group of a surface with boundary, and shows how hyperbolic geometry and surface topology are manifest in the abstract bounded cohomology of a free group.

Scl, sails and surgery

We establish a close connection between stable commutator length in free groups and the geometry of *sails* (roughly, the boundary of the convex hull of the set of integer lattice points) in integral polyhedral cones. This connection allows one to compute stable commutator length on certain infinite families of elements in a free group, those obtained by a line of surgeries on some fixed element in a free product of Abelian groups. Using this technology, one can show that the scl spectrum of a free group contains numbers congruent to every rational number mod Z, and contains well-ordered sequences of numbers with ordinal type ω^ω.



David Ellwood (CMI), Terry Tao (UCLA), Amanda Battese (CMI)

Perth, Western Australia

Program Overview

Schedule

July 23 - August 19,
Melbourne
Thursday, July 23, Danny Calegari: Surface subgroups from homology @ Melbourne University

Monday, August 3, Danny Calegari: Faces of the scl norm ball @ Melbourne University

Monday, August 10, Danny Calegari: Scl, sails, and surgery @ Melbourne University

Friday, August 14, Danny Calegari: Faces of the stable commutator length norm ball @ La Trobe University

August 31 - September 2,
Melbourne
Monday, August 31, Terry Tao:
Mathematical research and the internet @ Melbourne University

Tuesday, September 1, Terry Tao: Compressed sensing @ Royal Melbourne Institute of Technology

Wednesday, September 2, Terry Tao: Discrete random matrices @ Monash University

September 3 - 4,
at the University of Western Australia, Perth
Thursday, September 3, Terry Tao: The cosmic distance ladder

Friday, September 4, Terry Tao: Compressed Sensing

September 8 - 9,
Brisbane
Tuesday, September 8, Terry Tao: Cosmic Distance Ladder @ Queensland University of Technology

Wednesday, September 9, Terry Tao: Compressed sensing @ University of Queensland

September 15 - 18,
Sydney
Tuesday, September 15, Terry Tao: Compressed sensing @ Sydney University

Wednesday, September 16, Danny Calegari: Faces of the stable commutator length norm ball @ University of New South Wales

Wednesday, September 16, Terry Tao: Structure and randomness in the prime numbers @ University of New South Wales

Thursday, September 17, Terry Tao: Recent progress on the Kakeya problem @ Macquarie University

Thursday, September 17, Mohammed Abouzaid: Understanding hypersurfaces through tropical geometry @ Macquarie University

Friday, September 18, Danny Calegari: Faces of the stable commutator length norm ball @ Sydney University

Friday, September 18, Mohammed Abouzaid: A mirror construction for hypersurfaces in toric varieties @ Sydney University

September 21 - 24,
at the Australian National University, Canberra
Monday, September 21, Terry Tao: Recent progress on the Kakeya problem

Tuesday, September 22, Danny Calegari: Faces of the stable commutator length ball

Tuesday, September 22, Terry Tao: Structure and randomness in the prime numbers

Wednesday, September 23, Mohammed Abouzaid; Understanding hypersurfaces through tropical geometry

Wednesday, September 23, Terry Tao: Recent progress in additive prime number theory

Thursday, September 24, Danny Calegari: Faces of the stable commutator length norm ball

Thursday, September 24, Mohammed Abouzaid: A mirror construction for hypersurfaces in toric varieties

September 25,
at the University of Adelaide
Friday, September 25, Mohammed Abouzaid: Understanding hypersurfaces through tropical geometry

Friday, September 25, Danny Calegari: Faces of the stable commutator length norm ball

Friday, September 25, Terry Tao: The proof of the Poincaré conjecture

Septemer 28 - October 1,
AustMS 2009 Annual Conference, University of South Australia, Adelaide
Monday, Septemmber 28, Plenary lecture by Danny Calegari: Faces of the stable commutator length norm ball

Tuesday, September 29, Plenary lecture by Mohammed Abouzaid: Functoriality in homological mirror symmetry

Tuesday, September 29, Public lecture by Terry Tao: Structure and randomness in the prime numbers

Thursday, October 1, Plenary lecture by Terry Tao: The proof of the Poincaré conjecture

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Friday, October 2, Royal Institution of Australia Public Lecture by Terry Tao: The cosmic distance ladder

October 6 - 7,
at the University of Melbourne
Tuesday, October 6, Mohammed Abouzaid: Understanding hypersurfaces through tropical geometry

Wednesday, October 7, Mohammed Abouzaid, : String topology and the Fukaya category of cotangent bundles

Wednesday, October 7, Mohammed Abouzaid: A topological model for the Fukaya category of plumbings

CMI Workshops

Stringy Reflections on the LHC by Cumrun Vafa

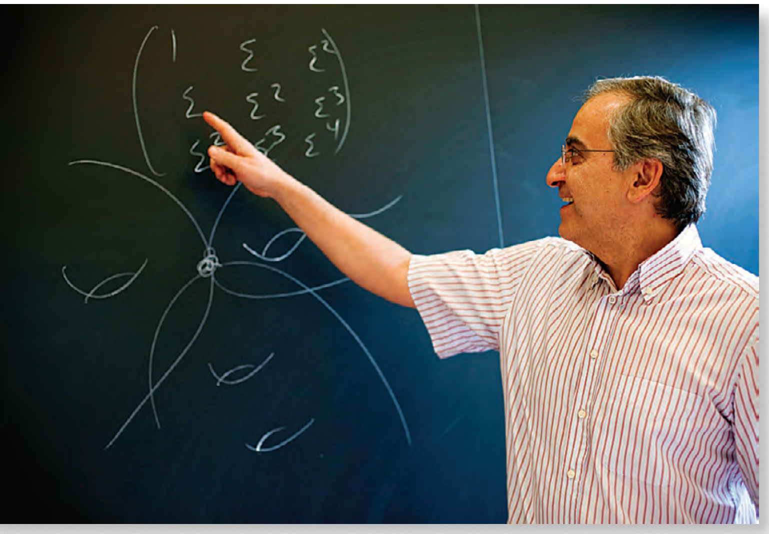


Photo courtesy of Stephanie Mitchell, Harvard University News Office

Organizers

James Carlson (CMI)
David Ellwood (CMI)
Cumrun Vafa (Harvard)
Herman Verlinde (Princeton)

One of the most exciting experiments of physics has just commenced at CERN. The Large Hadron Collider, LHC, collides two beams of protons moving with almost the speed of light. The center of mass energy is eventually targeted at ten TeV, about five times higher energy than the highest energy currently reached by any collider. By probing this energy region, we expect to be able to answer some of the most important questions that fundamental physics has tried to answer in the past forty years. A key missing ingredient in what is called the *standard model of particle physics* is a particle known as the *Higgs boson*. It is predicted to exist based on electroweak symmetry breaking. Moreover, all particles we know of are believed to receive their mass through their interactions with a condensate of the Higgs field.

While the discovery of the Higgs particle would be a spectacular confirmation of decades-long anticipation by theoretical physicists, that may not be the most exciting find of the LHC. Through astrophysical and cosmological observations, we know that the matter making up the universe consists mostly of unknown particles. Moreover, simple estimates of the energy range relevant for probing this so-called dark matter suggests that the LHC energy is roughly right for producing matter of this kind. The LHC may produce dark matter and thus solve a major puzzle of physics. The main theoretical question is: what do we expect this kind of matter to be?

Over the past few decades, string theory, with deep links to mathematics, has emerged as a prime candidate for unifying gravity with the other forces and for providing a consistent framework for a quantum theory of gravity. One important symmetry of string theory at the shortest distance scale is supersymmetry. This is a symmetry that relates bosons and fermions. In other words, for each particle there would exist its supersymmetric partner, whose spin differs by 1/2, but with otherwise exactly the same properties. We know that this symmetry cannot exist at the larger length scales at which we have performed experiments. For example, there is no partner of the electron ("selectron") which has the same mass and charge as the electron but which has spin zero. Thus supersymmetry, even if it is a true symmetry of nature at shortest distance scale, must be broken at longer distance scales.

Supersymmetry has also played a key role in connecting modern physics with mathematics. In particular, in the context of the topological field theories initiated by Witten, the concept of supersymmetry is a key ingredient. Supersymmetry yielded a deeper understanding of Donaldson invariants for smooth 4-manifolds, and had a significant impact on our understanding of enumerative geometry. Supersymmetry is thus aesthetically and mathematically a very rich structure. Since we know that this symmetry is not realized at the lowest energy scales, the main question for string theory is to explore at what scale this symmetry is broken. If it is broken at a very short distance (high energy) scale, there would be no leftover imprint of it at the scale at which the LHC will perform its experiments. This is a logical, though unfortunate possibility!

Program Overview

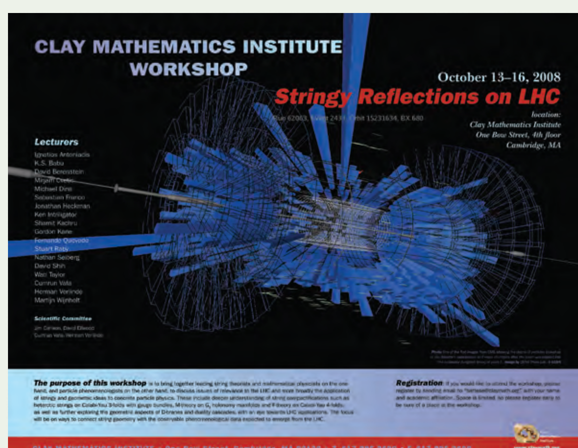
None the less, there are good reasons to speculate that supersymmetry plays a role at the energy scale of the LHC. One such reason is that unification of forces works more naturally in this context. Another reason has to do with the hierarchy problem. The hierarchy problem asks why the mass of the Higgs particle is so small compared to the fundamental mass scale in physics, that is, the Planck scale. Supersymmetry, while it would not by itself explain why the scale is so different, would explain why it is natural for this small mass scale to be stable with respect to quantum corrections. This would be the case as long as supersymmetry breaking occurs at sufficiently low energy scales. For these reasons, one of the most popular ideas pursued by particle theorists for models beyond the standard model has been the supersymmetric extension and its breaking at energies at the scale of the LHC.

Even if one assumes that supersymmetry plays a key role at the energy scales of the LHC, predictions of precisely what would be seen requires knowledge of how supersymmetry is broken. This symmetry breaking involves a choice of parameters—the choice can be viewed as the selection of a point on a manifold of 100+ dimensions! One can make various assumptions, as particle physicists have done to narrow the region for search, but still, typically, the remaining region is too wide to be viewed as capable to give a definitive prediction. The one prediction that essentially all such models make is the existence of a stable dark matter particle, the lightest of the supersymmetric particles. However, we need to narrow the choice of parameters for the supersymmetry-broken theory in order to make more specific predictions for the LHC, and thus to confirm or reject such theories.

At first sight, string theory seems not to help much in narrowing the parameter range for the search for supersymmetry breaking at low energy scales. However, with some mild assumptions (that the matter and gauge forces arise from tiny regions of internal compactification manifolds), some colleagues, in particular Jonathan Heckman, and I have made some surprisingly specific predictions. These predictions involve the study of the geometry of elliptically fibered Calabi-Yau 4-folds and the interpretation of the various singular loci in terms of physics. Using this work, we have come to the conclusion that if supersymmetry leaves an imprint at the LHC energy scale, the lightest supersymmetric particle is the gravitino (the supersymmetric partner of the graviton), with a mass one hundred times larger than that of the electron. Such a particle interacts too weakly to observe directly. Therefore, what is important is the nature of the next lightest particle. In our models, this particle turns out to be semi-stable, with a lifetime in the range of a second to an hour. There are two possibilities for this next particle. In most parameter ranges, it turns out to be a charged particle (known as the *stau*), which would leave a dramatic track in the LHC detectors. There is also the less likely possibility that this particle would be neutral (a particle known as *bino*). The bino can not be directly observed by the LHC, but can be discovered using conservation of energy. Other relations connecting mathematics, string theory, and physics were explored last fall at a workshop held at CMI ("Stringy Reflections on LHC").

The next few years may be among the most exciting for physicists' search for the fundamental laws of nature. The discovery of supersymmetry, if indeed it does occur, would be not only one of the most exciting discovery of a new principle of physics, but would also nicely mirror the important role supersymmetry has played in providing a bridge between physics and mathematics.

We will have to wait a few years and see!



Program Overview

CMI Workshops

Singularities by Jim Carlson

Jim Carlson

Heisuke Hironaka

In 1964 Heisuke Hironaka changed the world of mathematics by proving resolution of singularities in characteristic zero for varieties of arbitrary dimension. In dimension one the result is almost trivial, a consequence of the existence of normalization. In dimension two, informal proofs over the complex numbers date as far back as the work of Levi (1899), Chisini (1921), and Albanese (1924). Modern proofs for surfaces of characteristic zero were given by Walker (1935), Zariski (1939), and Abhyankar, who also did the non-imbedded positive characteristic p case (1956). Zariski established non-imbedded, characteristic zero resolution of singularities for varieties of dimension three in 1944.

Hironaka's theorem states that given an algebraic variety X defined over a field of characteristic zero, there is a smooth algebraic variety Y and a surjective map of Y to X that is an isomorphism over the complement of the set of non-smooth points of X. Moreover, he showed that one may choose Y and the map of Y to X such that the inverse image of the set of non-smooth points is a union of smooth hypersurfaces meeting transversally. For this work, Hironaka received the Fields Medal in 1970.

Almost overnight, Hironaka's theorem became one of the standard tools for work in algebraic geometry. To take just one example, Deligne's construction of a mixed Hodge structure on a quasi-projective variety relied on resolution of singularities.

Despite its status as a powerful and frequently applied theorem, few mathematicians had mastered the details of Hironaka's proof. However, over the the years, a sequence of papers—Villamayor 1989 and 1991, Bierstone and Milman (1991 + 1997), Encinas and Villamayor (1998), Encinas and Hauser (2002), Cutkosky (2004), Włodarczyk (2005), Kollár (2007)—led to a dramatically better understanding of the theorem. To give just one measure, Hironaka's original proof, occupying roughly 200 pages in the *Annals of Mathematics*, a very challenging read at the time, can now be presented to advanced graduate students using Kollár's book, which has about the same length. Kollár's book contains much preparatory material; the actual proof in its modern form for a working mathematician occupies about a tenth of the space of the original.This is progress!

There remained the challenge of proving resolution of singularities in positive characteristic. This has been done in dimension two (embedded case) and in dimension three with characteristic greater than five by Abhyankar. The general case for dimension three, that is, non-embedded resolution, is due to Cutkosky, who greatly simplified Abhyankar's proof, and by Cossart-Piltant, who removed the characteristic greater than five assumption. In addition, Cossart-Jannsen-Saito established embedded resolution of excellent surfaces of arbitrary codimension.

In an important different line of attack, Johan de Jong (1996) proved a weaker form of resolution in all dimensions and all characteristics: any variety X is dominated by a smooth variety Y of the same dimension by a generically finite map. For many applications this is sufficient; however, the map from Y to X need not be birational. Thus, despite much progress, the challenge of arbitrary characteristic and arbitrary dimension remains.

In the last few years, there have been renewed attempts to surmount this challenge. To this end, CMI organized a workshop during the week of September 22, 2008, that brought together many of the mathematicians who are working on the problem or have a strong interest in it: Dan Abramovich, Dale Cutkosky, Herwig Hauser, János Kollár, Heisuke Hironaka, James McKernan, Orlando Villamayor, and Jaroslaw Włodarczyk . A superset of this group met later at RIMS in Kyoto at a workshop organized by Shigefumi Mori, where the Kawanoue-Matsuki approach was also discussed. Professor Hironaka opened the CMI workshop with a talk at Harvard, and spoke energetically each morning at 9:30. Others gave talks as the workshop progressed, with organization quite fluid, and with much time for informal discussion.

The problem of resolution of singularities in arbitrary characteristic and arbitrary dimension, an area of renewed research interest, remains open. The article cited below gives some idea of the status of current efforts.

Reference

Herwig Hauser, on the Problem of Resolution of Singularities in Positive Characteristic (Or: A proof we are still waiting for). *Bull. Amer. Math. Soc.* 47, (2010),1-30.

CMI Workshops

Geometry of Outer Space

October 19 - 22, 2009

Organizers:
David Ellwood (CMI), Mladen Bestvina (Utah)

Speakers:
Mladen Bestvina (Utah), Yael Algom-Kfir (Utah), Martin Bridson (Oxford), Matt Clay (Allegheny College), Mark Feighn (Rutgers), Vincent Guirardel (Toulouse), Ursula Hamenstaedt (Bonn), Michael Handel (CUNY), Ilya Kapovich (Urbana-Champaign), Alexandra Pettet (Michigan), Juan Souto (University of Michigan), Karen Vogtmann (Cornell), Kasra Rafi (Oklahoma), Ruth Charney (Brandeis), Dan Margalit (Tufts)

Outer space is a contractible complex on which *Out* F^n, the group of outer automorphisms of a free group, acts properly. It is analogous to Teichmüller space, which plays a crucial role in the study of mapping class groups. The topology of outer space has been studied for the last twenty-five years and is now very well understood. However, our understanding of the geometry of outer space is lacking and has only begun to be investigated in the last couple of years. It lags behind the corresponding geometric understanding of Teichmüller space.

Of the recent developments, the following are significant: I. Kapovich-Lustig's proof that the length pairing between measured geodesic currents and R-trees is continuous (replacing the continuity of the intersection pairing between measured laminations), Bestvina-Feighn's construction of a Gromov-hyperbolic Out F^n-complex (replacing the hyperbolicity of the curve complex), Scott-Swarup-Guirardel's notion of intersection number and core for a pair of splittings, and its subsequent applications in the work of Clay-Pettet on detecting fully irreducible automorphisms (analogues of pseudo-Anosov mapping classes), Mosher-Handel's work on axes of fully irreducible automorphisms, Hamenstädt's work on lines of minima and her point of view of the boundary, using currents, and Algom-Kfir's theorem that axes of fully irreducible automorphisms are strongly contracting. Most of the above people were at the workshop, where they explained their work and points of view to the others.

The workshop provided a venue for geometric group theorists and Teichmüller geometers to meet in an informal setting that favored communication. Several of the lectures were attended by Curtis McMullen and his seminar participants. The sense of the meeting was optimistic. The hope is that the results cited above will be followed by breakthroughs that will give us a deep understanding of the geometry of outer space. A concrete goal is to prove the quasi-isometric rigidity of *Out* F^n. The analogous result for the mapping class group was achieved recently by Behrstock-Kleiner-Minsky-Mosher.

CMI Supported Conference

Singularities @ MIT: A Celebration of Richard Melrose's 60th Birthday

April 4, 2009 Massachusetts Institute of Technology

Peter Sarnak and Richard Melrose

Honoree and Organizers:

Richard Melrose (MIT)
Victor Guillemin (MIT)
Pierre Albin (MIT)
David Ellwood (CMI)

Speakers:

Jean-Michel Bismut (Paris-Sud XI)
Robert MacPherson (IAS)
Rafe Mazzeo (Stanford))
Peter Sarnak (Princeton)
Johannes Sjostrand (Bourgogne)
Isadore Singer (MIT)

This one-day event highlighted a topic that Richard Melrose extensively explored in his own research: the role played by singularities (and the desingularization of thereof by blowing-up techniques) in microlocal analysis, in non-linear PDE theory, and in differential geometry. The talks were an impressive testament to the fact that this "Melrose philosophy" has become more and more pervasive, not only on the theoretical side of this subject but also in down-to-earth applications. MacPherson's semi-popular talk on the role of singularities in material sciences provided beautiful examples of concrete applications of these techniques, and the talks of Sjostrand and Mazzeo, although a bit more technical, did so as well. At the other end of the spectrum were the talks of Bismut and Singer on the the role of these methods in quantum theory and of Sarnak on their role in analytic number theory.

This meeting also provided a much-needed opportunity for specialists in superficially different areas to communicate recent results. This was of benefit not only to the experts themselves, but especially to the many graduate students and young researchers who were able to attend. On the whole, the talks made a compelling case for "Singularities" as a viable new mathematical discipline not just "@MIT" but in the mathematics (and physics and material science) community at large.

CMI Supported Conference

Geometry and Physics: Atiyah80

The panel: Michael Atiyah, Peter Higgs, David Saxon, and Edward Witten

Organizers

David Ellwood (CMI)
Andrew Ranicki (University of Edinburgh)

Scientific Committee

Simon Donaldson FRS (Imperial College)
Nigel Hitchin FRS (Oxford)
Frances Kirwan FRS (Oxford)
Graeme Segal FRS (Oxford)

Photo courtesy of Thomas Koeppe

The "Atiyah 80" workshop was organized in honor of Sir Michael Atiyah's 80th birthday. It consisted of eleven lectures by Witten, Hitchin, Bridgeland, Kirwan, Hopkins, Segal, McDuff, Seidel, Vafa, Donaldson and Dijkgraaf on their latest research in geometry and physics, one historical lecture by Hirzebruch, and one panel discussion on the Higgs Boson involving Atiyah, Higgs, Saxon and Witten. The panel discussion was a joint event with the Royal Society of Edinburgh. The lecture by Hopkins announced the solution (with Hill and Ravenel) of the forty-five year-old Kervaire invariant problem, a major advance in homotopy theory. The meeting was a fitting tribute for Sir Michael Atiyah's 80th birthday, reflecting his enormous influence on current activities in both mathematics and physics. The conference benefited from the active participation of Sir Michael himself.

The lectures were of the high caliber to be expected from such a distinguished group of mathematicians. Hopkins' announcement of the solution of the Kervaire invariant problem and Donaldson's Kaehler-Einstein metric announcement were the scientific highlights of the conference.

The workshop offered a range of activities for all participants: apart from the lectures themselves there were ample opportunities for meetings and dicussions, as is made clear in the answers to the ICMS questionnaire. There was the opportunity to hear Atiyah himself speak at the panel discussion and at the conference dinner. An exhibition of posters about Atiyah prepared by Sebastià Xambo of Barcelona was displayed as well as biographical material about the late Raoul Bott and two related videos.

The conference website at http://www.maths.ed.ac.uk/~aar/atiyah80.htm includes videos of all the lectures, the audio recording of the Higgs panel discussion at the Royal Society of Edinburgh, and much other Atiyah80-related material. This includes a photo of the 6 bottles of whisky of total age 80 years which were presented to Sir Michael by the conference! The University Communication and Marketing Department made a short YouTube film "Great minds honour maths maestro" which gives a faithful snapshot of the conference atmosphere, and includes interviews with Atiyah himself, Bourguignon, Witten and M. Singer.

Thanks to the generosity of the various sponsors (EPSRC, LMS, EMS, NSF, the Clay Mathematics Institute, and Trinity College, Cambridge) it was possible for the workshop to support not only the speakers but also a large contingent of the collaborators (such as I.Singer) and mathematical descendants of Atiyah, and many postdoctoral researchers and doctoral students.

The main workshop was followed by a successful one-day "Atiyah80plus" meeting at the headquarters of ICMS in 14 India Street (supported by the Roberts Fund for Transferable Skills) which was organized by the Edinburgh postgraduate students, with talks by Frances Kirwan and visiting postdoctoral researchers. But perhaps the most remarkable feature of the workshop was that the sun shone every day.

CMI Supported Conference

Dynamical Numbers: Interplay between Dynamical Systems and Number Theory

July 20 - 24, 2009, Max Planck Institute for Mathematics, Bonn

The International conference "Dynamical Numbers - Interplay between Dynamical Systems and Number Theory" was the main event of the Max Planck Institute (MPI) for Mathematics program "Dynamical Numbers: Interplay between Dynamical Systems and Number Theory Dynamical Systems" (May 1- July 31, 2009).

The theory of dynamical systems is currently one of the most vibrant areas of mathematical research, born from H. Poincaré's "Les Méthodes nouvelles de la Mécanique Céleste" at the end of nineteenth century. Dynamical systems theory draws on methods from many branches of mathematics (algebra, analysis, geometry, and topology) and has arisen in attempting to formulate an adequate description of phenomena in the world around us. It provides the theoretical framework for many models in physics, biology, economics, and other fields. It also contributes tools for solving problems in other fields of mathematics. Finally, the perspective of dynamical systems theory leads one to ask completely new questions.

Organizers

David Ellwood (CMI), Sergiy Kolyada (Institute of Mathematics of the NASU, Ukraine & MPIM), Yuri Manin (MPIM & Northwestern University), Martin Moeller (MPIM), Pieter Moree (MPIM), Don Zagier (MPIM & Collège de France)

While the classical branches of dynamical systems theory — ergodic theory, topological dynamics, and low-dimensional, smooth, and complex dynamics — have grown in importance, completely new areas have appeared of late, including algebraic and arithmetic dynamics.

The goal of the MPI conference was to stimulate communication and cooperation among participants in the various branches of this very diverse field. The eighty participants and twenty-six lectures by leading experts in the core areas of dynamical systems, number theory, and closely related areas gave a panoramic view of current research.

The main subjects of the lectures were: asymptotic geometric analysis and (topological) transformation groups; arithmetic dynamics; polynomials and pointwise ergodic theorems; Bernoulli convolutions; actions of Polish groups; low-dimensional dynamics: graph theory, rotation theory, smooth interval dynamics, area preserving diffeomorphisms and time-one maps on surfaces; complex and real dynamics; interval exchange transformations and translation flow; billiards; leafwise cohomology of algebraic Anosov diffeomorphisms; symbolic dynamics; multifractal analysis and Diophantine approximations; dynamics and moduli spaces; invariant measures and Littlewood's conjecture; Moebius number systems, flows on manifolds; translation surfaces and Abelian differentials, symbolic representations of toral automorphisms, noncommutative Mahler measures; representations of integers; statistical properties of dynamical systems; transfer operators for geodesic flows and Hecke operators; transfer operators for Anosov diffeomorphisms; shift operators on buildings and noncommutative spaces; topological orbit equivalence; and theory of entropy and chaos.

Galois Representations, University of Hawai'i at Manoa

by Zachary A. Kent

Organizers

Brain Conrad (Stanford)
David Ellwood (CMI)
Mark Kisin (Harvard)
Chris Skinner (Princeton)

The University of Hawaiʻi at Mānoa in Honolulu, Hawaiʻi provided an exquisite location for the 2009 Clay Mathematics Institute Summer School. The aim of the school was to provide an overview of the ideas and applications of the theory of Galois representations. Many advances in number theory within the last fifteen years (such as the solutions of the Shimura-Taniyama conjecture, the Sato-Tate conjecture and Serre's conjecture, as well as decisive progress on the Fontaine-Mazur conjecture and main conjectures for modular forms) have relied heavily upon advances in the theory of Galois representations. For example, such advances have enabled the local and global aspects of modularity lifting theorems to be extended far beyond the traditional 2-dimensional case over the rational numbers, and have led to generalizations of the "classical" theory of p-adic modular forms in a way that makes more effective use of representation theory and geometry to obtain results on the arithmetic of L-values.

The program was built around three foundational courses:

- p-adic Hodge Theory by Olivier Brinon and Brian Conrad
- Deformations of Galois Representations and Modular Forms by Mark Kisin and Jacques Tilouine
- Iwasawa Theory and Automorphic Applications by Joël Bellaïche and Chris Skinner

These courses were supplemented by several mini-courses:

- Proofs of p-adic Comparison Theorems by Fabrizio Andreatta

Galois Representations

by Zachary A. Kent continued

- Introduction to the p-adic Langlands Correspondence by Matthew Emerton
- Construction of Galois Representations by Sug Woo Shin

Our motivating example comes from the Birch and Swinnerton-Dyer conjecture, which asserts that the rank of an elliptic curve E over a number field F is encoded in a p-adic representation $\rho : G_F \to \mathrm{GL}_2(\mathbb{Z}_p)$ where p is a rational prime and G_F is the absolute Galois group of F. For any prime $\wp$ in F we may choose an embedding of algebraic closures $\overline{F} \hookrightarrow \overline{F_\wp}$ and we may restrict to get a representation $\rho_\wp := \rho|_{G_{F_\wp}}$ that encodes local information about E at $\wp$. Birch and Swinnerton-Dyer believed that the rank of E is determined by the behavior at $s = 1$ of the meromorphically continued L-function with Euler product over primes $\wp$ in F of good reduction. For each p, the Galois representation on the Tate module of E encodes all of the Euler factors at primes $\wp$ of good reduction that are also unramified (for $\wp \nmid p$). For this reason, the theory of p-adic representations of Galois groups is very useful for studying the arithmetic of L-functions. On the other hand, for primes $\wp$ of good reduction such that $\wp \mid p$, we must use deep results of p-adic Hodge theory to replace the property of being unramified at $\wp$ with the property of being crystalline at $\wp$.

The p-adic variant of Hodge theory has its origins in Serre and Tate's study of Tate modules for abelian varieties with good reduction over p-adic fields and the concept of a Hodge-Tate representation. Brian Conrad began with this example in the first foundational course and continued by discussing the deep result of Faltings relating p-adic cohomology to Hodge cohomology, i.e., the comparison isomorphism C_{HT}. The coefficient ring for B_{HT} for Faltings' isomorphism is called a *period ring*. Another example given was the concept of étale ϕ-modules, which comes from the Fontaine-Wintenberger theory of norm fields and classifies all p-adic representations of G_K for any field K of characteristic p.

Olivier Brinon and Brian Conrad then took turns in presenting the more general theory developed by Fontaine and his coworkers. They focused on the construction of several other period rings B_{dR} (de Rham), B_{cris} (crystalline), and B_{st} (semi-stable), for passing between pairs of p-adic cohomology theories, i.e., the comparison isomorphisms C_{dR}, C_{cris}, and C_{st}. Next, they introduced p-divisible groups as a testbed for various ideas in p-adic Hodge theory, while filtered (ϕ, N)-modules were discussed as a way to deal with "bad reduction." They then considered integral p-adic Hodge theory making use of rigid-analytic geometry. They did this for the purpose of studying Galois deformation theory with artinian coefficents, which requires a finer theory (introduced by Fontaine and Laffaille with more recent developments by Breuil and Kisin). In this theory, p-adic vector spaces are replaced by lattices or torsion modules. Finally, (ϕ, Γ)-modules were introduced as an improvement over the theory of étale ϕ-modules, since they provide a classification of all p-adic representations of the entire Galois group G_K. Several applications of (ϕ, Γ)-modules were then discussed, including the proof of overconvergence of p-adic representations and recent developments in the p-adic Langlands correspondence for GL_2.

In his mini-course, Fabrizio Andreatta presented a new proof of Fontaine's crystalline conjecture, which is the comparison isomorphism C_{cris} relating p-adic étale cohomology and crystalline cohomology. After studying the underlying sheaf theory of a special topology of Faltings, a new cohomology theory was defined for sheaves of periods that was then used to compute the cohomologies appearing in C_{cris} via the theory of almost étale extensions.

Returning to our motivating example from earlier, Andrew Wiles developed techniques for proving that various representations $\rho : G_F \to \mathrm{GL}_2(\mathbb{Z}_p)$ not initially related to modular forms actually come from them in a specific way. He does this by deforming ρ, e.g., we may consider the deformation $\widetilde{\rho} : G_F \to \mathrm{GL}_2(\mathbb{Z}_p[[x]])$ that recovers ρ at $x = 0$ and is unramified at all but finitely many primes of F. Understanding deformations of $\rho|_{G_{F_\wp}}$ when $\wp|p$ was a key idea behind his technique. Some of the most important improvements of Wiles' technique were discussed by their creator, Mark Kisin, in the second foundational course. He began by discussing deformations of pro-finite group representations and pseudo-representations. He showed, in particular, that one may study deformation theory by considering deformations of trace functions. Next, Kisin talked about representability of deformations and global deformation rings. Finishing off by making use of integral p-adic Hodge theory, he introduced flat deformations which arise from finite flat group schemes over a ring of integers.

Jacques Tilouine continued the second foundational course from the dual perspective of deformation theory: quaternionic and Hilbert modular forms and their Galois representations. He began by discussing Shimura varieties and later introduced quaternionic and Hilbert forms. After proving several results about eigenforms and eigenvarieties, he focused on the Jacquet-Langlands correspondence, which he repeatedly used to prove various theorems throughout his lectures. Tilouine then moved on to Galois representations and local versus global deformations. Recalling Kisin's work throughout his lectures, Tilouine finished by discussing the Taylor-Wiles-Kisin method.

The third foundational course was dedicated to Iwasawa theory and automorphic applications. For his part, Chris Skinner focused on Iwasawa theory and special values of L-functions. After an introduction to the major results in a basic course in algebraic number theory, Skinner sketched the ideas behind a proof for the main conjecture of Iwasawa theory for a large class of elliptic curves. Along the way, he was able to show that the vanishing at a certain point of an L-function associated with eigenforms of even integer weight and trivial level means that the associated Selmer group (Galois cohomology group related to the Galois representation under consideration) is nonzero.

Joël Bellaïche discussed two automorphic applications as part of the third foundational course. The first application was the Bloch-Kato

Galois Representations

by Zachary A. Kent continued

conjecture, which describes the mysterious relationship between the algebraic and analytic objects associated to a geometric representation of V of G_F. The algebraic object is called the Bloch-Kato Selmer group, $H^1_f(G_F, V)$, and the analytic object is a meromorphic L-function of a complex variable s, $L(V, s)$. The second application was Ribet's lemma, a tool to construct non-trivial extensions of Galois representations. After presenting the lemma and its proof, Bellaïche discussed some generalizations of Ribet's lemma for pseudo-representations. He ended his lectures by proving the fundamental theorems by Taylor and Rouquier-Nyssen.

A special case of the global Langlands correspondence was the focus of Sug Woo Shin's mini-course. In his lecture, he discussed some ideas centered on the conjecture in the case of the group GL_m, and he explained how to use the cohomology of varieties over number fields to construct the predicted Galois representations. Viewing things locally, the focus of Matthew Emerton's mini-course was to provide an introduction to the p-adic Langlands correspondence. It follows that the p-adic local Langlands correspondence coupled with a certain local-global compatibility conjecture implies most cases of the Fontaine-Mazur conjecture for two-dimensional odd representations of $G_{\mathbb{Q}}$.

All lecturers included topics of interest to advanced students, but also took care to provide concrete examples that were accessible to non-experts.

Selected Articles by Research Fellows

Mohammed Abouzaid

"Framed bordism and Lagrangian embeddings of exotic spheres," arXiv:0812.4781.

"Homological mirror symmetry for the four-torus," with Ivan Smith. To appear in Duke Mathematical Journal.

Spiridon Alexakis

"Hawking's local rigidity theorem without analyticity," with A. Ionescu and S. Klainerman. Accepted for publication at Geometric and Functional Analysis.

"The decomposition of global conformal invariants IV: A proposition on local Riemannian invariants." Accepted for publication at Advances in Mathematics.

Artur Avila

"The ten martini problem," with Svetlana Jitomirskaya. Annals of Mathematics 170 (2009), 303-342.

"Combinatorial rigidity for unicritical polynomials," with Jeremy Kahn, Mikhail Lyubich, and Weixiao Shen. Annals of Mathematics 170 (2009), 783-797.

Soren Galatius

"Monoids of moduli spaces of manifolds," with O. Randal-Williams. Submitted. arXiv:0905.2855.

"Madsen-Weiss for geometrically minded topologists," with Y. Eliashberg, S. Galatius, and N. Mishachev. Submitted. arXiv:0907.4226.

Adrian Ioana

"Non-orbit equivalent actions of F^n." Annales scientifiques de l'ENS 42, fascicule 4 (2009), 675-696.

"Ergodic subequivalence relations induced by a Bernoulli cction," with Ionut Chifan. Accepted for publication in Geometric and Functional Analysis.

"Subequivalence relations and positive-definite functions," with TodorTsankov and Alekos S. Kechris. Groups, Geometry, and Dynamics, Volume 3, Issue 4, (2009), 579-625.

Davesh Maulik

"Néron-Severi groups under specialization," with B. Poonen and C. Voisin. To appear in Les Annales Scientifiques de l'École Normale Supérieure.

Sophie Morel

"Cohomologie d'intersection des variétès modulaires de Siegel, suite" (2008).

"Note sur les polynomes de Kazhdan-Lusztig" (2009).

Sam Payne

"Cayley decompositions of lattice polytopes and upper bounds for h*-polynomials," with Christian Haase and Benjamin Nill. J. reine angew. Math. 637 (2009), 207–216.

"A tropical proof of the Brill-Noether theorem," with Filip Cools, Jan Draisma, and Elina Robeva. Preprint. arXiv:1001.2774.

Sucharit Sarkar

"A combinatorial description of knot Floer homology," with Ciprian Manolescu and Peter Ozsváth. Annals of Mathematics 169, (2009), 633-660.

David Speyer

"The multidimensional cube recurrence," with Andre Henriques Advances in Mathematics, Volume 223, Issue 3, (2010), 1107-1136.

"Sortable elements in infinite Coxeter groups," with Nathan Reading. To appear in Transactions of the AMS.

Teruyoshi Yoshida

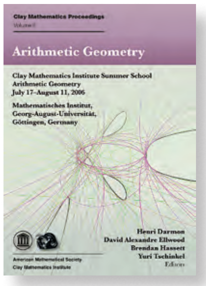
"Local class field theory via Lubin-Tate theory." Annales de la Faculté des Sciences de Toulouse, Ser. 6, 17-2 (2008), 411-438. (math.NT/0606108)

Xinyi Yuan

"Calabi-Yau theorem and algebraic dynamics," with WShou-wu Zhang.

"On volumes of arithmetic line bundles II."

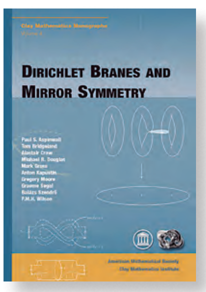
Books and Videos



Arithmetic Geometry;

Proceedings of the 2006 CMI Summer School at Gottingen.
Editors: Henri Darmon, David Ellwood, Brendan Hassett, Yuri Tschinkel. CMI/AMS 2009, 562 pp., http://www.claymath.org/publications/Arithmetic_Geometry.

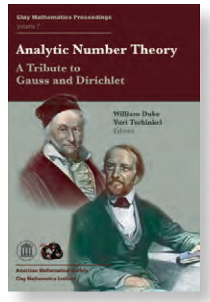
This book is based on survey lectures given at the 2006 CMI Summer School at the Mathematics Institute of the University of Gottingen. It will introduce readers to modern techniques and outstanding conjectures at the interface of number theory and algebraic geometry.



Dirichlet Branes and Mirror Symmetry

Editors: Michael Douglas, Mark Gross. CMI/AMS 2009, 681 pp., http://www.claymath.org/publications/Dirichlet_Branes.

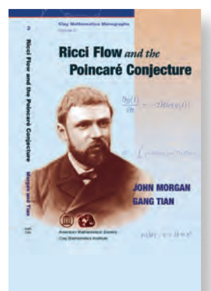
The book first introduces the notion of Dirichlet brane in the context of topological quantum field theories, and then reviews the basics of string theory. After showing how notions of branes arose in string theory, it turns to an introduction to the algebraic geometry, sheaf theory, and homological algebra needed to define and work with derived categories. The physical existence conditions for branes are then discussed, culminating in Bridgeland's definition of stability structures. The book continues with detailed treatments of the Strominger-Yau-Zaslow conjecture, Calabi-Yau metrics and homological mirror symmetry, and discusses more recent physical developments.



Analytic Number Theory:

A Tribute to Gauss and Dirichlet.
Editors: William Duke, Yuri Tschinkel. CMI/AMS, 2007, 265 pp., www.claymath.org/publications/Gauss_Dirichlet.

This volume contains the proceedings of the Gauss–Dirichlet Conference held in Göttingen from June 20–24 in 2005, commemorating the 150th anniversary of the death of Gauss and the 200th anniversary of Dirichlet's birth. It begins with a definitive summary of the life and work of Dirichlet by J. Elstrodt and continues with thirteen papers by leading experts on research topics of current interest within number theory that were directly influenced by Gauss and Dirichlet.



Ricci Flow and the Poincaré Conjecture

Authors: John Morgan, Gang Tian. CMI/AMS, 2007, 521 pp., www.claymath.org/publications/ricciflow.

This book presents a complete and detailed proof of the Poincaré conjecture. This conjecture was formulated by Henri Poincaré in 1904 and has remained open until the recent work of Grigory Perelman. The arguments given in the book are a detailed version of those that appear in Perelman's three preprints.

The Millennium Prize Problems

Editors: James Carlson, Arthur Jaffe, Andrew Wiles. CMI/AMS, 2006, 165 pp., www.claymath.org/publications/Millennium_Problems.

This volume gives the official description of each of the seven problems as well as the rules governing the prizes. It also contains an essay by Jeremy Gray on the history of prize problems in mathematics.

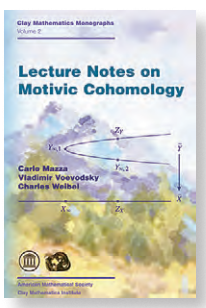
Books and Videos

Floer Homology, Gauge Theory, and Low-Dimensional Topology;

Proceedings of the CMI 2004 Summer School at Rényi Institute of Mathematics, Budapest.

Editors: David Ellwood, Peter Ozsváth, András Stipsicz, Zoltán Szábo. CMI/AMS, 2006, 297 pp., www.claymath.org/publications/Floer_Homology.

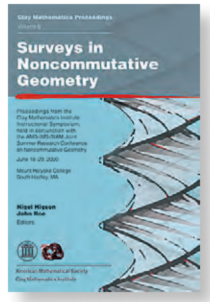
This volume grew out of the summer school that took place in June of 2004 at the Alfréd Rényi Institute of Mathematics in Budapest, Hungary. It provides a state-of-the-art introduction to current research, covering material from Heegaard Floer homology, contact geometry, smooth four-manifold topology, and symplectic four-manifolds.



Lecture Notes on Motivic Cohomology

Authors: Carlo Mazza, Vladimir Voevodsky, Charles Weibel. CMI/AMS, 2006, 210 pp., www.claymath.org/publications/Motivic_Cohomology.

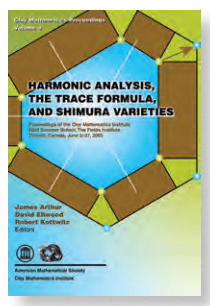
This book provides an account of the triangulated theory of motives. Its purpose is to introduce the reader to motivic cohomology, to develop its main properties, and finally to relate it to other known invariants of algebraic varieties and rings such as Milnor K-theory, étale cohomology, and Chow groups.



Surveys in Noncommutative Geometry

Editors: Nigel Higson, John Roe. CMI/AMS, 2006, 189 pp., www.claymath.org/publications/Noncommutative_Geometry.

In June of 2000, a summer school on noncommutative geometry, organized jointly by the American Mathematical Society and the Clay Mathematics Institute, was held at Mount Holyoke College in Massachusetts. The meeting centered around several series of expository lectures that were intended to introduce key topics in noncommutative geometry to mathematicians unfamiliar with the subject. Those expository lectures have been edited and are reproduced in this volume.



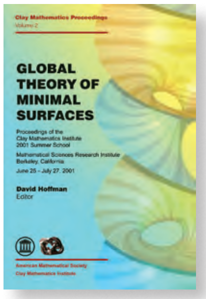
Harmonic Analysis, the Trace Formula and Shimura Varieties;

Proceedings of the 2003 CMI Summer School at Fields Institute, Toronto.

Editors: James Arthur, David Ellwood, Robert Kottwitz. CMI/AMS, 2005, 689 pp., www.claymath.org/publications/Harmonic_Analysis.

The subject of this volume is the trace formula and Shimura varieties. These areas have been especially difficult to learn because of a lack of expository material. This volume aims to rectify that problem. It is based on the courses given at the 2003 Clay Mathematics Institute Summer School. Many of the articles have been expanded into comprehensive introductions, either to the trace formula or to the theory of Shimura varieties, or to some aspect of the interplay and application of the two areas.

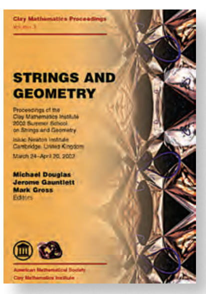
Books and Videos



Global Theory of Minimal Surfaces

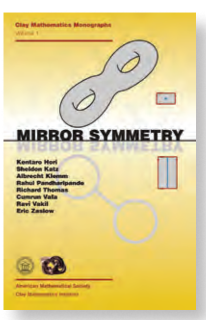
Proceedings of the 2001 CMI Summer School at MSRI.
Editor: David Hoffman. CMI/AMS, 2005, 800 pp., www.claymath.org/publications/Minimal_Surfaces.

This book is the product of the 2001 CMI Summer School held at MSRI. The subjects covered include minimal and constant-mean-curvature submanifolds, geometric measure theory and the double-bubble conjecture, Lagrangian geometry, numerical simulation of geometric phenomena, applications of mean curvature to general relativity and Riemannian geometry, the isoperimetric problem, the geometry of fully nonlinear elliptic equations, and applications to the topology of three-manifolds.



Strings and Geometry

Proceedings of the 2002 CMI Summer School held at the Isaac Newton Institute for Mathematical Sciences, UK.
Editors: Michael Douglas, Jerome Gauntlett, Mark Gross. CMI/AMS, 376 pp., paperback, ISBN 0-8218-3715-X. List: $69. AMS Members: $55. Order code: CMIP/3. To order, visit www.ams.org/bookstore.



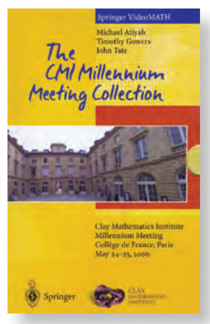
Mirror Symmetry

Authors: Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Ravi Vakil.
Editors: Cumrun Vafa, Eric Zaslow. CMI/AMS, 929 pp., hardcover, ISBN 0-8218-2955-6. List: $124. AMS Members: $99. Order code: CMIM/1. To order, visit www.ams.org/bookstore.

Strings 2001

Authors: Atish Dabholkar, Sunil Mukhi, Spenta R. Wadia. Tata Institute of Fundamental Research.
Editor: American Mathematical Society (AMS), 2002, 489 pp., paperback, ISBN 0-8218-2981-5. List: $74. AMS Members: $59. Order code: CMIP/1. To order, visit www.ams.org/bookstore.

Video Cassettes



The CMI Millennium Meeting Collection

Authors: Michael Atiyah, Timothy Gowers, John Tate, François Tisseyere.
Editors: Tom Apostol, Jean-Pierre Bourguignon, Michele Emmer, Hans-Christian Hege, Konrad Polthier. Springer VideoMATH, Clay Mathematics Institute, 2002.

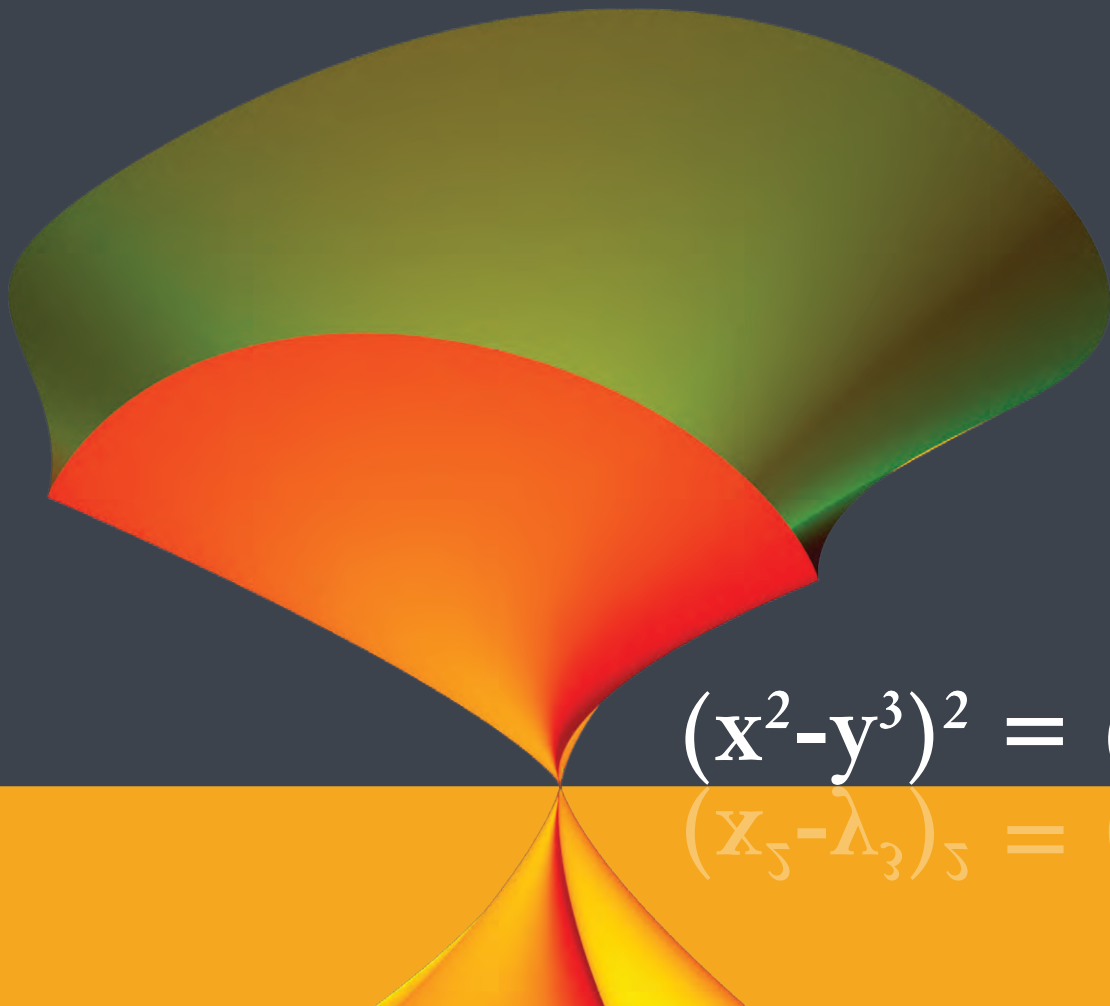
Box set consists of four video cassettes: The CMI Millennium Prize Problems, a lecture by Michael Atiyah; and The Millennium Prize Problems, a lecture by John Tate. VHS/NTSC or PAL. ISBN 3-540-92657-7. List: $119, EUR 104.95. To order, visit www.springer-ny.com (in the United States) or www.springer.de (in Europe).

These videos document the Paris meeting at the Collége de France where CMI announced the Millennium Prize Problems. The videos are for anyone who wants to learn more about these seven grand challenges in mathematics.

Videos of the 2000 Millennium event are available online and in VHS format from Springer-Verlag. To order the box set or individual tapes, visit www.springer.com.

2010 Institute Calendar

Date	Event	Location
January 4-March 31, 2010	Galois Trimester at the Institut Henri Poincare (IHP)	Paris, France
January 18-22 and March 8-12, 2010	Senior Scholar Pierre-Louis Lions at Issac Newton Institute (INS) Stochastic Partial Differential Equations (SPD)	Cambridge
January 1-May 31, 2010	Senior Scholar Tomasz Mrowka at MSRI - Homology Theories of Knots and Links	MSRI
January 11-May 21, 2010	Senior Scholar Peter Ozsvath at MSRI - Homology Theories of Knots and	MSRI
March 8-11, 2010	CMI Workshop on Macdonald Polynomials and Geometry	Cambridge
June 2-6, 2010	Number Theory & Representation Theory in Honor of Dick Gross	Harvard
June 7-9, 2010	Clay Research Conference at the Institut Henri Poincare (IHP)	Paris, France
June 14-July 3, 2010	ICTP Summer School on Hodge Theory	ICTP Trieste, Italy
June 27-July 17, 2010	Senior Scholar Ingrid Daubechies at PCMI - The Mathematics of Image Processing	PCMI
July 11-August 7, 2010	CMI Summer School 2010 Probability & Statistical Physics in Two (and More) Dimensions	Buzios, Brazil
July 26-August 6, 2010	Winter School on Topics in Noncommutative Geometry at Departamento de Matemàtica	Universidad de Buenos Aires
August 2-9, 2010	Conference in Honor of the 70th birthday of Endre Szemerédi	Budapest, Hungary



$$(x^2-y^3)^2 = (z^2-y^2)^3$$

Workshops at CMI

CMI holds a limited number of Workshops at its One Bow Street office in Cambridge, Massachusetts. Although the format is flexible, these are typically of three to six days duration with eight to sixteen participants. Participants are fully funded. Workshops in topics undergoing rapid development can be organized with little lead time. Please contact Jim Carlson (jcarlson@claymath.org) or David Ellwood (ellwood@claymath.org) to discuss a proposal.

Workshops to date have been held on the following topics: Goodwillie Calculus (Haynes Miller and Jack Morava), Algebraic Statistics and Computational Biology (Bernd Sturmfels, Lior Pachter, Seth Sullivant and Jim Carlson), Eigenvarieties (David Ellwood, Barry Mazur and Richard Taylor), Moduli Spaces of Vector Bundles (David Ellwood, Emma Previato and Montserrat Teixidor-i-Bigas), Hopf Algebras and Props (David Ellwood, Jean-Louis Loday, and Richard Stanley), Recent Developments in Symplectic Topology (Denis Auroux, Victor Guillemin, Tomasz Mrowka, and Katrin Wehrheim), Recent Developments in Higher-dimensional Algebraic Geometry (in conjunction with the 2007 Clay Research Conference), Solvability and Spectral Instability (Nils Dencker, David Ellwood, and Maciej Zworski), Computational Arithmetic Geometry (Jim Carlson, David Harvey, Kiran Kedlaya, and William Stein), Rational Curves and Diophantine Problems over Function Fields (David Ellwood, Brendan Hassett, Johan de Jong, Jason Starr, and Yuri Tschinkel), Shrinking Target Properties (Jayadev Athreya, David Ellwood, Anish Ghosh, and Dmitry Kleinbock), K3s: Modular Forms, Moduli, and String Theory (Jim Carlson, David Ellwood, Davesh Maulik, and Rahul Pandharipande), Automorphic Forms in Moduli Problems of Schottky and Brill-Noether Type (David Ellwood, Emma Previato and Leon Takhtajan), The Foundations of Algebraic Geometry: Grothendieck's EGA Unbound (Jim Carlson, David Ellwood, Joe Harris, and David Smyth), Stringy Reflections on LHC (Jim Carlson, David Ellwood, Cumrun Vafa, and Herman Verlinde), Geometry and Physics of the Landau-Ginzburg Model (David Ellwood, Yongbin Ruan, and Tyler Jarvis).

See www.claymath.org/programs/cmiworkshops