Boundaries of Teichmüller spaces and end-invariants for hyperbolic 3-manifolds

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Abstract. We study two boundaries for the Teichmüller space of a surface Teich(S) due to Bers and Thurston. Each point in Bers' boundary is a hyperbolic 3-manifold with an associated geodesic lamination on S, its *end-invariant*, while each point in Thurston's is a *measured* geodesic lamination, up to scale. We show that when $\dim_{\mathbb{C}}(\text{Teich}(S)) > 1$ the end-invariant is not a continuous map to Thurston's boundary modulo forgetting the measure with the quotient topology. We recover continuity by allowing as limits maximal measurable sub-laminations of Hausdorff limits and enlargements thereof.

Contents

1. Introduction

In celebrated boundaries for Teichmüller space due to Bers and Thurston, geodesic laminations arise in natural ways:

- A point M in Bers' boundary, a hyperbolic 3-manifold, has an associated geodesic lamination $\mathcal{E}(M)$ that has been *pinched*. The lamination $\mathcal{E}(M)$ is an invariant of the quasi-isometry class $[M]$ of M.
- A point $[\mu]$ in Thurston's boundary, a measured lamination μ up to scale, records the asymptotic stretching of divergent hyperbolic metrics $X_i \to [\mu]$. Its support $|\mu|$ is a geodesic lamination.

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Thurston's *ending lamination conjecture* predicts that the map $[M] \mapsto \mathcal{E}(M)$ from quasi-isometry classes in Bers' boundary to the quotient of Thurston's boundary by forgetting the measure is an injection. In other words, if one knows the lamination $\mathcal{E}(M)$, one knows the manifold M up to quasi-isometry. The map $\mathcal{E}$ gives a bijection between dense subsets: the dense family of *maximal cusps* M (a maximal family of simple closed curves is pinched in M) is mapped by $\mathcal{E}$ to the dense set of *maximal partitions* of S by simple closed curves (which are analogous to rational points of S^1). Thus, given Thurston's conjecture, it is natural to ask whether $\mathcal{E}$ is a homeomorphism. Or, as a starting point, how do sequences $\mathcal{E}(M_n)$ behave under limits $M_n \to M$?

In this paper we show $\mathcal{E}$ has the following continuity properties:

I. $\mathcal{E}$ is strictly lower-semi-continuous in the quotient topologies,
II. $\mathcal{E}$ is continuous in a new *end-invariant topology*, based on the Hausdorff topology, which predicts new information about its limiting values, and
III. $\mathcal{E}$ cannot have a continuous inverse in the end-invariant topology, nor do Hausdorff limits completely encode the limiting end-invariant in general.

To state our results more precisely, we review terminology.

Let S be an oriented surface, closed for simplicity, and let $Q(X, Y)$ denote the quasi-Fuchsian *Bers simultaneous uniformization* of the pair of surfaces $(X, Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$ (where $\overline{S}$ is S with the reverse orientation). Such uniformizations sit in the closed subset $AH(S)$ of the *representation variety*

$$\mathcal{V}(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))/\text{conjugation}$$

consisting of representations that are discrete and faithful.

The map $Q\colon \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S}) \to AH(S)$ is a homeomorphism onto its image, the quasi-Fuchsian space $QF(S) \subset AH(S)$. Fixing Y in the second factor gives the *Bers slice* $B_Y \cong \mathrm{Teich}(S)$ of $QF(S)$. Bers proved B_Y has compact closure in $AH(S)$, giving rise to a *Bers compactification* $\overline{B_Y}$ and a *Bers boundary* ∂B_Y.

The measured laminations $\mathcal{ML}(S)$ on S are a natural completion of the isotopy classes of essential simple closed curves on S with positive real weights. Projectivizing, one obtains a sphere $\mathcal{PL}(S) = \mathcal{ML}(S) - \{0\}/\mathbb{R}_+$ of *projective measured laminations* with which Thurston compactifies $\mathrm{Teich}(S)$. On any hyperbolic surface X, each measured lamination μ determines a *geodesic lamination*, a closed subset of X foliated by geodesics, as its *support* $|\mu|$.

Representations $\rho \in AH(S)$ are in bijection with *marked* hyperbolic 3-manifolds $(f\colon S \to M)$ up to homotopy, where $M = \mathbb{H}^3/\rho(\pi_1(S))$ and $f_* = \rho$. Thurston associates an *end-invariant* $\mathcal{E}(M)$ to each $M \in \partial B_Y$, namely, the geodesic lamination consisting of all non-peripheral parabolics and laminations on which any measure has 'length-zero' in M (see §2). Since any such geodesic lamination is *measurable* (it arises in the quotient of Thurston's boundary by forgetting the measure), $\mathcal{E}$ gives a mapping

$$\mathcal{E}\colon \partial B_Y \to \mathcal{PL}(S)/|.|.$$

The lamination $\mathcal{E}(M)$ is an invariant of the marked quasi-isometry class $[M]$ of M. Letting $\partial B_Y/\mathrm{qi}$ denote the quotient of ∂B_Y by marking preserving quasi-isometry, $\mathcal{E}$ descends to a mapping $\mathcal{E}\colon \partial B_Y/\mathrm{qi} \to \mathcal{PL}(S)/|.|$ which we also denote by $\mathcal{E}$.

Our first theorem is the following.

Theorem 1.1. *The mapping $\mathcal{E}$ is strictly lower-semi-continuous in the quotient topologies on domain and range.*

Here, lower-semi-continuity means:

$$\text{for} \quad [M_n] \to [M] \quad \text{any limit} \quad \mathcal{E}_\infty \quad \text{of} \quad \{\mathcal{E}([M_n])\} \quad \text{satisfies} \quad \mathcal{E}_\infty \subset \mathcal{E}([M]).$$

Strict lower-semi-continuity means there exists $M_n \to M$ for which the final containment is proper (see theorem 4.1).

Note that maximal families of pairwise disjoint, essential simple closed curves are dense in $\mathcal{PL}(S)/|.|$. These are the images under $\mathcal{E}$ of *maximal cusps*: 3-manifolds $M \in \partial B_Y$ for which the curves in such a maximal family are parabolic. The invariant $\mathcal{E}(M)$ determines the maximal cusp M up to isometry. The question of the continuity properties of $\mathcal{E}$ is then motivated by

THEOREM 1.2 (McMullen). *Maximal cusps are dense in ∂B_Y.*

Theorem 1.1 contrasts the behavior of maximal families as measures and as parabolics in the passage to limits.

Before recovering continuity, we give a characterization of the laminations that can arise in the image of $\mathcal{E}$. A measurable lamination $\nu \in \mathcal{PL}(S)/|.|$ *fills* a compact surface S if for any essential simple closed curve α on S that is not parallel to ∂S, α intersects ν. Decompose ν into the union $\nu = P \sqcup E$ of its simple closed curve components P and its infinite *minimal* components E for which every leaf is infinite and dense in its component. We say ν *relatively fills* S if any component ν' of E fills the subsurface of $S - P$ that it meets. Let $\mathcal{EL}(S)$ be the quotient of the quotient $\mathcal{PL}(S)/|.|$ obtained assigning to $\nu \in \mathcal{PL}(S)/|.|$ the lamination $\widehat{\nu} \in \mathcal{PL}(S)/|.|$ given by by adding to ν the minimal set of simple closed curves required to obtain a lamination that relatively fills S.

Compactness theorems for Thurston's *pleated surfaces* show that $\mathcal{E}$ takes values in $\mathcal{EL}(S)$ (§3). Given $\nu \in \mathcal{EL}(S)$, we may use theorem 1.1 to find an $M \in \partial B_Y$ for which $\mathcal{E}(M) = \nu$: pinching P and families of simple closed curves approximating E to cusps, we extract a limit M with $\mathcal{E}(M) = \nu$. This gives a new proof[1] of:

THEOREM 1.3. *The mapping $\mathcal{E}$ is a surjection onto $\mathcal{EL}(S)$.*

We introduce a new topology on $\mathcal{EL}(S)$: the *end-invariant topology* is the topology of convergence for which

(*) *$\nu_n \to \nu$ if for any subsequence ν_{n_j} converging to λ_H in the Hausdorff topology, ν contains the maximal measurable sub-lamination η of λ_H.*

(The end-invariant topology, like the quotient topologies, is non-Hausdorff). Then we obtain the following strengthening of theorem 1.1 (theorem 5.3):

THEOREM 1.4. *The mapping $\mathcal{E}$ is continuous from the quotient topology on $\partial B_Y/\mathrm{qi}$ to $\mathcal{EL}(S)$ with the end-invariant topology.*

In general, given a convergent sequence $M_n \to M$ in ∂B_Y, the end-invariants $\mathcal{E}(M_n)$ need not converge in the Hausdorff topology. Theorem 1.4 forces the measurable sub-laminations of any pair Hausdorff limits of $\mathcal{E}(M_n)$ into alignment.

The main techniques in this paper are developed in [**Br1**] where we prove a bi-continuity theorem for the *lengths* of measured laminations realized by pleated surfaces in hyperbolic 3-manifolds. The end invariant $\mathcal{E}(M)$ is the zero-set of this length function when M is fixed.

These questions relate to the following

[1]K. Ohshika gave a proof of surjectivity of $\mathcal{E}$ in [**Ohs1**] but his proof assumed a special case of the main result of [**Br1**]. This special case was claimed by Thurston but had not appeared.

CONJECTURE 1.5 (Thurston). *The map* $\mathcal{E}\colon \partial B_Y/\mathrm{qi} \to \mathcal{EL}(S)$ *is a bijection.*

One may speculate as to whether $\mathcal{E}$ gives a homeomorphism in any reasonable topology on $\mathcal{EL}(S)$. Theorems 1.2 and 1.4 show $\mathcal{E}$ cannot have a continuous inverse in the end-invariant topology (§7).

Convergence in a Bers compactification. The possibility of pinching in the conformal boundary of M means the end-invariant topology must allow for the constant sequence to enlarge in the limit. We record this extra information by considering maximal families of disjoint simple closed curves on $\partial M - Y$ whose lengths in M and on Y are in small ratio. Indeed, given $M_n \to M$ in the Bers compactification $\overline{B_Y}$ there is a family $\Pi(M_n)$ of such curves so that $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ is a geodesic lamination and

$$\lim_{n\to\infty} \max_{\gamma \in \Pi(M_n)} \frac{\mathrm{length}_{M_n}(\gamma)}{\mathrm{length}_Y(\gamma)} = 0.$$

Then we prove the following (see corollary 6.3):

THEOREM 1.6. *The laminations* $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ *converge to* $\mathcal{E}(M)$ *in the end-invariant topology.*

In the case when each $\mathcal{E}(M_n)$ is maximal (a maximal partition, say) it is reasonable to ask whether given the maximal measurable sub-lamination η of the Hausdorff limit λ_H of $\mathcal{E}(M_n)$, the lamination $\widehat{\eta}$ is the full end-invariant $\mathcal{E}(M)$. Though the answer is yes in many cases, we conclude this paper with a negative answer to this question in general (see theorem 7.1):

THEOREM 1.7. IMPLICIT CUSPS *Let* γ *be an essential simple closed curve in* S*. Then for any other essential simple closed curve* α *in* $S - \gamma$*, there are maximal partitions* $C_n \to \lambda_H$ *in the Hausdorff topology and associated maximal cusps* $M(C_n) \to M$ *in* ∂B_Y *for which:*

1. γ *is the maximal measurable sub-lamination of* λ_H*, and*
2. α *lies in* $\mathcal{E}(M)$*.*

The curve α is an "implicit cusp" forced by 3-dimensional hyperbolic geometry that, somewhat surprisingly, goes undetected by the Hausdorff topology. The example producing theorem 1.7 reveals a new geometric phenomenon that complicates the relationship between hyperbolic surfaces and the 3-manifolds they parameterize.

History and references. The density of maximal cusps in Bers' boundary is proven by McMullen in [**Mc2**]. Whether or not appropriate quotients of Bers' and Thurston's boundaries are homeomorphic is asked by McMullen in [**Mc3**]. For informative discussions of the end-invariant see [**Mc4**] and [**Min2**].

In general, we allow S to be compact with nonempty boundary. Indeed, when $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) = 1$, Y. Minsky has shown (see [**Min3**]) that that $\mathcal{E}$ is a homeomorphism from ∂B_Y to $\mathcal{PL}(S)$ (passing to quotients is redundant as the support $|\mu|$ of any measured lamination $\mu \in \mathcal{ML}(S)$ admits a unique transverse measure up to scale, and Minsky proves that $\mathcal{E}(M)$ determines M up to isometry). Note that in this setting $\mathcal{E}(M)$ is always connected, while when $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) > 1$, the invariant $\mathcal{E}(M)$ can be disconnected.

Thurston introduces pleated surfaces and lengths of laminations in [**Th1**], [**Th2**], and [**Th4**]. Various versions of Thurston's length function are discussed in [**Th4**], [**Bon3**] and [**Ohs2**]; we prove a general bi-continuity theorem (see theorem 2.3) in [**Br1**] where the key lemmas on nearly-straight train tracks employed in the proof of theorem 1.4 ([**Br1**, Lem. 5.2, Cor. 5.3]) also appear.

We have chosen to work in the Bers slice to avoid certain technicalities that arise in more general deformation spaces of hyperbolic 3-manifolds. We remark that work of J. Anderson and R. Canary [**AC**] reveals a different type of possible discontinuity in the analogous end-invariant mapping for general deformation spaces (see [**Min3**, §12]). We plan to merge these two perspectives in a sequel.

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2. Preliminaries

Let S be an oriented compact topological surface of negative Euler characteristic. We allow S to have non-empty boundary; let $\text{int}(S) = S - \partial S$ denote its interior.

Teich(S). The *Teichmüller space* $\text{Teich}(S)$ is the space of finite-area hyperbolic surfaces X equipped with homeomorphisms $f\colon \text{int}(S) \to X$ such that

$$(f\colon \text{int}(S) \to X) \sim (g\colon \text{int}(S) \to Y)$$

if there is an isometry $\phi\colon X \to Y$ so that $\phi \circ f \simeq g$.

The topology on $\text{Teich}(S)$ is induced by the natural distance $d(X, Y)$ obtained by taking the infimum K over all k for which there is a k-bi-Lipschitz diffeomorphism ϕ homotopic to $g \circ f^{-1}$ and setting $d(X, Y) = \log(K)$. The Teichmüller space is homeomorphic to an open ball and carries a natural complex structure of dimension $\dim_{\mathbb{C}}(\text{Teich}(S)) = 3g - 3 + n$, where S has genus g with n boundary components.

***AH*(S).** Let $\mathcal{D}(S)$ denote the space of discrete faithful representations $\rho\colon \pi_1(S) \to \text{Isom}^+(\mathbb{H}^3)$ so that $\rho(\gamma)$ is parabolic for each peripheral element $\gamma \in \pi_1(S)$ (i.e. γ is boundary-parallel), with the compact-open topology, or the topology of *algebraic convergence.* Let

$$AH(S) = \mathcal{D}(S)/\text{Isom}^+(\mathbb{H}^3)$$

be its quotient by conjugation.

By a theorem of Thurston and Bonahon [**Th1**, Ch. 9] [**Bon1**] $M = \mathbb{H}^3/\rho(\pi_1(S))$ is a complete hyperbolic manifold homeomorphic to $\text{int}(S)\times\mathbb{R}$. The complete hyperbolic manifold M is prolonged to its *Kleinian manifold* $\overline{M}$ by adding its conformal boundary ∂M: namely, the quotient of the domain $\Omega(M) \subset \widehat{\mathbb{C}}$ where $\rho(\pi_1(S))$ acts properly discontinuously.

The set of hyperbolic 3-manifolds M marked by homotopy equivalences $(f\colon S \to M)$ up to marking-preserving isometry is in bijection with conjugacy classes of representations $\rho \in AH(S)$ via the association $f \mapsto f_*$. Thus we will often speak of $AH(S)$ as a space of marked hyperbolic manifolds and write $M \in AH(S)$, assuming an implicit marking homotopy equivalence $(f\colon S \to M)$.

One may formulate algebraic convergence in this context: $\{(f_n\colon S \to M_n)\}$ converges to $(f\colon S \to M)$ if for any compact set $K \subset M$ there are smooth, marking-preserving homotopy equivalences $q_n\colon M \to M_n$ that converge to a local isometry on K in the C^∞ topology (see [**Mc5**, §3.1]; we refer the reader to [**Mc5**], [**Th1**], or [**Br2**] for details about hyperbolic 3-manifolds and Kleinian groups).

***QF*(*S*).** By a theorem of Bers [**Bers1**] there is unique *quasi-Fuchsian* manifold $Q(X,Y) \in AH(S)$ interpolating between any pair of hyperbolic surfaces $(X,Y) \in \text{Teich}(S) \times \text{Teich}(\overline{S})$ in its conformal boundary. Given $Y \in \text{Teich}(S)$, the *Bers slice*

$$B_Y = \{Q(X,Y) \ : \ X \in \text{Teich}(S)\}$$

is an embedded copy of Teich(S) in $AH(S)$. The embedding depends on Y, but for any Y the slice B_Y is precompact in $AH(S)$. One obtains a *Bers compactification* $\overline{B_Y}$ by forming the closure, and an associated *Bers boundary* for Teichmüller space as its boundary ∂B_Y (see also [**KT**], [**Mc5**], or [**Bers2**]).

$\mathcal{ML}(S)$. Let $\mathcal{S}$ be the set of isotopy classes of essential non-peripheral simple closed curves on S. The *geometric intersection number*

$$i \colon \mathcal{S} \times \mathcal{S} \to \mathbb{Z}_{\geq 0}$$

counts the minimal number $i(\alpha,\beta)$ of intersections of curves in distinct isotopy classes (α,β) in $\mathcal{S}\times\mathcal{S}$ and takes the value zero on the diagonal.

Attaching a positive real weight to each isotopy class, let

$$\iota \colon \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}^{\mathcal{S}}$$

be defined by

$$\langle \iota(t\gamma)\rangle_\alpha = t i(\alpha,\gamma).$$

Then we define the *measured laminations* $\mathcal{ML}(S) = \overline{\iota(\mathbb{R}_+ \times \mathcal{S})}$ by taking the closure of the image (note that weighted simple closed curves are naturally dense in $\mathcal{ML}(S)$). The intersection number extends to a symmetric continuous function $i \colon \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}_{\geq 0}$ so that $i(s\,\alpha, t\,\beta) = s \cdot t(i(\alpha,\beta))$ for $\alpha,\beta \in \mathcal{S}$ and $s,t \in \mathbb{R}_{\geq 0}$ [**Bon1**, Prop. 4.5].

The measured lamination space $\mathcal{ML}(S)$ is a cell of the same real dimension as Teich(S). The *projective measured laminations* $\mathcal{PL}(S) = \mathcal{ML}(S) - \{0\}/\mathbb{R}_+$ form a sphere of one dimension lower. The sphere $\mathcal{PL}(S)$ is Thurston's boundary for Teichmüller space - the topology on Thurston's compactification $\text{Teich}(S) \sqcup \mathcal{PL}(S)$ is determined by the conditions that Teich(S) is open in $\text{Teich}(S) \sqcup \mathcal{PL}(S)$ and $X_n \to [\mu] \in \mathcal{PL}(S)$ if and only if

$$\frac{\text{length}_{X_n}(\alpha)}{\text{length}_{X_n}(\beta)} \to \frac{i(\mu,\alpha)}{i(\mu,\beta)}$$

for any pair α and β in $\mathcal{S}$ for which $i(\mu,\beta) \neq 0$. (For more on measured and projective laminations, and Thurston's compactification see [**FLP**], [**Th1**], or [**Bon2**]).

Subsurfaces. A *subsurface* is a compact 2-submanifold of S. An *essential* subsurface $T \subset S$ is a subsurface so that each curve in ∂T is homotopically essential. Given an essential subsurface $T \subset S$, let $\mathcal{S}(T) \subset \mathcal{S}$ be isotopy classes of simple closed curves in $\mathcal{S}$ isotopic into T that are non-peripheral in T. Then $\mathcal{ML}(T)$ is naturally a closed subspace of $\mathcal{ML}(S)$.

$\mathcal{GL}(S)$. Given $X \in \text{Teich}(S)$, a *geodesic lamination* λ on X is a closed subset of X that admits a decomposition into complete simple geodesics called *leaves* of λ. The set of geodesic laminations $\mathcal{GL}(X)$ on X is a compact subspace of the space of closed subsets $\text{Cl}(X)$ in the Hausdorff topology.

Via a natural *circle at infinity* for S, geodesic laminations are canonically associated to the surface S and can be *realized* geodesically on any $X \in \mathrm{Teich}(S)$ via its implicit marking (see [**Bon2**], [**Fl**], or [**CEG**, §4.1]). Thus we will speak of a point $\lambda \in \mathcal{GL}(S)$, which determines a geodesic lamination on any particular hyperbolic surface $X \in \mathrm{Teich}(S)$. Given $\lambda \subset \mathcal{GL}(S)$, let $S(\lambda) \subset S$ be the essential subsurface obtained by realizing λ on $(f\colon S \to X) \in \mathrm{Teich}(S)$ and pulling back by f^{-1} the smallest subsurface with geodesic boundary containing λ.

A measured lamination $\mu \in \mathcal{ML}(S)$ determines a *transverse measure* on a geodesic lamination $|\mu|$. The geodesic lamination $|\mu|$ is called the *support* of μ. A geodesic lamination ν is *measurable* if there is some $\mu \in \mathcal{ML}(S)$ for which $\nu = |\mu|$; ν admits a transverse measure of full support.

Given $\lambda, \nu \in \mathcal{GL}(S)$, the notation $\lambda \subset \nu$ will mean that λ is a sub-lamination of ν, while the notation $\lambda \cap \nu$ will refer to any common sublamination of λ and ν together with the set of transverse intersections of leaves of λ and ν, well defined on any hyperbolic surface $X \in \mathrm{Teich}(S)$.

Pleated surfaces. Let $(f\colon S \to M) \in AH(S)$ and let $\lambda \in \mathcal{GL}(S)$ be a geodesic lamination. We say λ is *realizable* in M if there is a hyperbolic surface $X \in \mathrm{Teich}(S)$, and a *path-isometry*[2] $g\colon X \to M$, compatible with markings on X and M, so that $g|_\lambda$ is a local isometry. If g is totally geodesic on the complement of some geodesic lamination λ' containing λ, the triple (g, X, M) is called a *pleated surface* in M, and we say the pleated surface *realizes* λ. A measured lamination $\mu \in \mathcal{ML}(S)$ is *realizable* in M if its support $|\mu|$ is realizable. Any realizable lamination can be realized by a pleated surface.

Let $\mathcal{PS}(f)$ denote the set of all pairs (g, X), where $(\phi\colon S \to X) \in \mathrm{Teich}(S)$, and $g\colon X \to M$ is a pleated surface with $f \simeq g \circ \phi$. Let $\mathcal{PS}_{\mathrm{np}}(f) \subset \mathcal{PS}(f)$ be the subset for which $f_*(\gamma)$ is parabolic only if γ is a peripheral element of $\pi_1(S)$.

We topologize $\mathcal{PS}(f)$ by the Teichmüller distance on the underlying surfaces and the topology of uniform convergence on compact sets on the pleated mappings. In other words, $(g_n, X_n) \to (g, X)$ if there are marking-preserving bi-Lipschitz diffeomorphisms $q_n\colon X \to X_n$ with bi-Lipschitz constant tending to 1 so that the composition $g_n \circ q_n$ converges uniformly on compact subsets to g. Then we have the following compactness result due to Thurston (see [**CEG**, 5.2.18]):

Theorem 2.1 (Thurston). Pleated Surfaces Compact *Let $(f\colon S \to M) \in AH(S)$, and let $K \subset M$ be a compact subset. Then the set of all $(g, X) \in \mathcal{PS}_{\mathrm{np}}(f)$ with the property that $g(X) \cap K \neq \varnothing$ is compact.*

Also relevant is the following theorem which we restate in a form useful to us.

Theorem 2.2 (Thurston). Limits Realized *Let $\{(g_n, X_n)\} \subset \mathcal{PS}_{\mathrm{np}}(f)$ converge to (g, X) and let (g_n, X_n) realize convergent measured laminations $\mu_n \to \mu$. Then (g, X) realizes μ.*

(The theorem is a direct consequence of [**CEG**, 5.3.2]).

Lengths of laminations. Given $X \in \mathrm{Teich}(S)$, any isotopy class $\gamma \in \mathcal{S}$ has a well defined *length* by taking the arclength $\ell_X(\gamma^*)$ of its geodesic representative γ^*. By a theorem of Thurston and Bonahon (see [**Th4**] [**Bon1**, Prop. 4.5]) there is a unique continuous function

$$\mathrm{length}\colon \mathrm{Teich}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

[2] The map g sends geodesic arcs in X to rectifiable arcs in M of the same length.

that restricts to $\mathbb{R}_+ \times \mathcal{S}$ by

$$\text{length}_X(t\gamma) = t\ell_X(\gamma^*).$$

Let $\mathfrak{R} \subset AH(S) \times \mathcal{ML}(S)$ denote the set of pairs (M, μ) such that μ is realizable in M. We define the *length function*

$$\text{length}\colon \mathfrak{R} \to \mathbb{R}$$

by setting $\text{length}_M(\mu) = \text{length}_X(\mu)$ where $g\colon X \to M$ is any pleated surface realizing $|\mu|$ (the length in M does not depend on the realizing pleated surface; see [**Th4**] [**Bon4**]).

When μ is not realizable in M, proper sub-laminations may still be realizable. Define the projection map

$$\mathrm{R}_M\colon \mathcal{ML}(S) \to \mathcal{ML}(S)$$

to be the identity on laminations realizable in M and to associate to any non-realizable lamination μ the maximal sub-lamination $\mathrm{R}_M(\mu)$ of μ that is realizable in M.

Then we have the following from [**Br1**]:

Theorem 2.3. Length Continuous *The function*

$$\underline{\text{length}}\colon AH(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

given by $(M, \mu) \to \text{length}_M(\mathrm{R}_M(\mu))$ *is continuous.*

In particular, we have the following corollary:

Corollary 2.4. *Let pairs* $\{(M_n, \mu_n)\}$ *converge to* (M, μ) *in* $AH(S) \times \mathcal{ML}(S)$ *so that* $\underline{\text{length}}_{M_n}(\mu_n) \to 0$. *Then* $\mathrm{R}_M(\mu) = 0$.

In other words, if μ lies in $\mathcal{ML}(S)_+$, the non-zero elements of $\mathcal{ML}(S)$, and $\underline{\text{length}}_M(\mu) = 0$, then each component of μ is non-realizable in M.

The end invariant $\mathcal{E}(M)$. We make the following definition.

Definition 2.5. *Let* $M \in \partial B_Y$ *be a point in a Bers' boundary. Then its* end invariant $\mathcal{E}(M)$ *is the union of all connected geodesic laminations* λ *such that for some* $\mu \in \mathcal{ML}(S)_+$ *we have,*

$$\lambda = |\mu| \quad \textit{and} \quad \underline{\text{length}}_M(\mu) = 0.$$

By a theorem of Thurston and Bonahon (the *geometric tameness* of M [**Th1**], [**Bon1**]), $\mathcal{E}(M)$ lies in $\mathcal{PL}(S)/|.|$; i.e. $\mathcal{E}(M)$ is itself a measurable geodesic lamination.

Notation: Throughout, the notation $n \gg 0$ will mean 'all n sufficiently large.' Unless otherwise stated, constants will depend only on S.

3. Surjectivity onto measurable laminations that relatively fill

In this section, we reprise implications of compactness of pleated surfaces on the basic structure of $\mathcal{E}(M)$ (this theory is developed in [**Th1**, Ch. 9]) and go on to give a characterization of laminations that arise in the image of $\mathcal{E}$.

Decomposing laminations. A *partition* P of S is a collection $P \subset \mathcal{S}$ of distinct isotopy classes of pairwise-disjoint, essential, non-peripheral, simple closed curves on S. A *maximal partition* is a partition that cannot be enlarged. The partition P

determines a collection of essential subsurfaces in its complement as the complement of pairwise embedded open annular neighborhoods of each curve in P. Let $S - P$ denote their union, abusing notation.

Each measurable lamination ν (i.e. $\nu \in \mathcal{PL}(S)/|.|$) admits a decomposition

$$\nu = P(\nu) \sqcup E(\nu)$$

where $P(\nu) \subset \mathcal{S}$ is a partition, and each component of $E(\nu)$ is infinite and *minimal*: each leaf of $E(\nu)$ is bi-infinite and dense in its component. A general geodesic lamination λ decomposes into its maximal measurable sub-lamination $\nu \subset \lambda$ and a finite collection of bi-infinite leaves each end of which is either asymptotic to ν or to a puncture of S (see [**Otal**, §A]).

The measurable lamination ν *fills* S if for each $\alpha \in \mathcal{S}$, and any measure $\mu \in \mathcal{ML}(S)$ with $|\mu| = \nu$ we have either $i(\mu, \alpha) > 0$ or α is peripheral in S.

Generalizing, we make the following definition.

DEFINITION 3.1. *The measurable lamination ν* relatively fills S *if for each component $\nu' \subset E(\nu)$, ν' fills the subsurface component of $S - P(\nu)$ in which it lies.*

We define $\mathcal{EL}(S) \subset \mathcal{PL}(S)/|.|$ to be the subset of laminations that relatively fill S. Each measurable ν has an *implicit partition* $\widehat{P}(\nu)$: this is the minimal partition containing $P(\nu)$ so that $E(\nu) \sqcup \widehat{P}(\nu)$ is a lamination that relatively fills S. There is a natural projection

$$\mathcal{PL}(S)/|.| \to \mathcal{EL}(S) \quad \text{given by} \quad \nu \mapsto E(\nu) \sqcup \widehat{P}(\nu);$$

let $\widehat{\nu} = E(\nu) \sqcup \widehat{P}(\nu)$ (see figure 1).

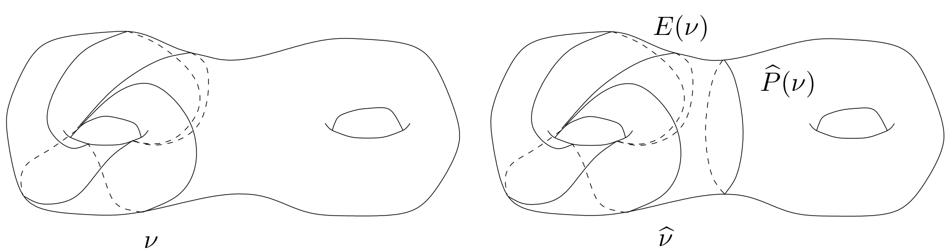

Figure 1. Adding the implicit partition $\widehat{P}(\nu)$.

In this section we prove the following:

THEOREM 3.2. *The map $\mathcal{E}$ is a surjection onto $\mathcal{EL}(S)$.*

We first prove $\mathcal{E}$ is well-defined as a map to $\mathcal{EL}(S)$.

LEMMA 3.3. *For any $M \in \partial B_Y$, the end-invariant $\mathcal{E}(M)$ relatively fills S.*

Proof: Let $(f : S \to M)$ be the implicit marking for M, and let $\mathcal{E}(M) = P \sqcup E$ be the decomposition of $\mathcal{E}(M)$ into its sets of parabolics P and infinite minimal components E. If $\mathcal{E}(M)$ does not relatively fill S, then for some connected sub-lamination $\nu \subset E$ lying in a connected component T of $S - P$, there is a simple closed curve $\gamma \in \mathcal{S}(T)$ in the implicit partition for ν that is non-peripheral in T. It follows that γ is not parabolic in M and is therefore realizable (see [**Th1**, §9.7], [**CEG**, Thm. 5.3.11]).

Let $t_n c_n \to \mu$, be a sequence of weighted simple closed curves converging to a measured lamination μ with support $\nu = |\mu|$ so that $i(\gamma, c_n) = 0$. There is a sequence of pleated surfaces $(g_n, X_n) \in \mathcal{PS}_{\mathrm{np}}(f|_T)$ realizing $\gamma \cup c_n$. Since (g_n, X_n) all realize γ, a subsequence converges to $(g, X) \in \mathcal{PS}_{\mathrm{np}}(f|_T)$ by theorem 2.1. By theorem 2.2, the limit realizes ν, a contradiction. Thus γ either intersects ν or lies in P, so ν relatively fills S.

∎

(A similar argument appears in [**Br2**, Thm. 4.7]).

Proof: *(of theorem 3.2).* Let $\nu \in \mathcal{EL}(S)$. Then there is a measured lamination $\mu \in \mathcal{ML}(S)$ so that $|\mu| = \nu$. Let $\Pi = P(\nu)$, let $E(\nu) = \nu_1 \sqcup \ldots \sqcup \nu_k$, and let

$$S - \Pi = S_1 \sqcup \ldots \sqcup S_k \sqcup T_1 \sqcup \ldots \sqcup T_s$$

denote the collection of subsurfaces of S determined up to isotopy as the complement of small pairwise embedded open annular neighborhoods of the curves in Π, so that ν_j lies in $\mathcal{GL}(S_j)$, $j = 1, \ldots, k$. Let $\mu_j \subset \mu$ denote the measured sub-lamination so that $|\mu_j| = \nu_j$.

For each j, let $\{c_{j,n}\} \subset \mathbb{S}$ be simple closed curves in $\mathbb{S}(S_j)$ so that for positive real weights $t_{j,n}$ we have $t_{j,n} c_{j,n} \to \mu_j$ as $j \to \infty$. Letting $\mu_\Pi \subset \mu$ be the measure determined by μ on Π (i.e. $|\mu_\Pi| = \Pi$), the unions

$$\xi_n = \mu_\Pi \bigcup \left(\sqcup_{j=1}^{k} t_{j,n} c_{j,n} \right)$$

are measured laminations so that $\xi_n \to \mu$ in $\mathcal{ML}(S)$.

A maximal partition $\mathcal{P}$ of S determines Fenchel-Nielsen length and twist coordinates

$$(\mathrm{length}_\gamma(X), \mathrm{twist}_\gamma(X)) \in \mathbb{R}_+^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

for $X \in \mathrm{Teich}(S)$, where $\gamma \in \mathcal{P}$ (see e.g. [**IT**]). Given a subset $P \subset \mathcal{P}$, the *pinching deformation* along P is the family of Riemann surfaces $X_t \in \mathrm{Teich}(S)$, $t \to 0$, determined by setting the coordinates

$$\mathrm{length}_\gamma(X_t) = t\,\mathrm{length}_\gamma(X)$$

for each $\gamma \in P$ and leaving all other coordinates unchanged. Then the pinching deformation along P determines a path $Q(X_t, Y)$ in B_Y that converges to a limit $M \in \partial B_Y$ with $\mathcal{E}(M) = P$ (see [**Ab**], [**Mc6**, Thm. 9.5]).

Let $M_n \in \partial B_Y$ be obtained from the quasi-Fuchsian manifold $Q(X, Y)$ by performing the pinching deformation along the collection

$$P_n = |\xi_n| = \Pi \bigcup \left(\sqcup_{j=1}^{k} c_{j,n} \right)$$

on X. For given r, and for each M_n let $W_n \in \mathrm{Teich}(T_r)$ denote the corresponding conformal boundary component of M_n. With respect to a fixed maximal partition $\mathcal{P}_T$ of $\cup_r T_r$, the Fenchel-Nielsen coordinates for W_n are the limiting Fenchel-Nielsen coordinates for X_t along $\mathcal{P}_T \cap T_r$. Hence, they do not depend on n and W_n is constant; we set $W_n = W$.

We have

$$\underline{\mathrm{length}}_{M_n}(\xi_n) = 0$$

for all n. By continuity of $\underline{\mathrm{length}}$ [**Br1**, Thm. 7.1], we have

$$\underline{\mathrm{length}}_M(\mu) = 0.$$

Since $\mu \in \mathcal{ML}(S)_+$, it follows that each component of μ is non-realizable in M. Thus $\nu = |\mu|$ is a sub-lamination of $\mathcal{E}(M)$.

Let $f\colon S \to M$ denote the implicit marking on M, and let $\pi_1(T_r)$ denote the subgroup of $\pi_1(S)$ induced by inclusion $T_r \subset S$ after choosing a basepoint in T_r. Since $\mathcal{E}(M)$ relatively fills S by lemma 3.3, to see that $\nu = \mathcal{E}(M)$ it suffices to show that the cover $\widetilde{M}(r)$ of M corresponding to $f_*(\pi_1(T_r))$ is quasi-Fuchsian (every lamination is realizable in a quasi-Fuchsian manifold, see [**Th1**, Prop. 8.7.7] [**CEG**, Thm. 5.3.11]).

Let $f_n\colon S \to M_n$ denote the implicit markings on M_n. For fixed r, the cover of M_n corresponding to $(f_n)_*(\pi_1(T_r))$ is a quasi-Fuchsian manifold $Q(W, Z_n) \in QF(T_r)$. The cover $\widetilde{Y}_r$ of Y corresponding to $\pi_1(T_r)$ (which is no longer of finite type) admits a holomorphic inclusion into Z_n, which is a contraction of the Poincaré metric by the Schwarz lemma. Thus, there is a pair of simple closed curves α and β in $\mathcal{S}(T_r)$ that *bind* T_r (i.e. $i(\alpha,\gamma)+i(\beta,\gamma) > 0$ for any $\gamma \in \mathcal{S}(T_r)$) and have uniformly bounded length in Z_n. Such a bound guarantees that Z_n range in a compact subset of Teich(T_r) (see e.g. [**Th4**, Prop. 2.4] [**Ker**]) so $Q(W, Z_n)$ converges to a quasi-Fuchsian manifold $Q(W, Z_\infty)$. Thus $\widetilde{M}(r)$ is quasi-Fuchsian, since it is the limit of $Q(W, Z_n)$.

It follows that $\nu = \mathcal{E}(M)$, and the theorem is proven.

■

4. Lower-semi-continuity

From now on, we view $\mathcal{E}$ as a map from quasi-isometry classes $[M] \in \partial B_Y/\text{qi}$ to the quotient $\mathcal{EL}(S)$ of $\mathcal{PL}(S)$ under the projection $[\mu] \mapsto \widehat{|\mu|}$. In this section we investigate the behavior of $\mathcal{E}$ in the quotient topologies on domain and range.

THEOREM 4.1. *Let* $\dim_{\mathbb{C}}(\text{Teich}(S)) > 1$*. Then the mapping* $\mathcal{E}$ *is strictly lower-semi-continuous in the quotient topologies.*

Again, 'lower-semi-continuity' has the interpretation:

$$\text{(4.1)}\quad \text{Given } [M_n] \to [M] \text{ any limit } \mathcal{E}_\infty \text{ of } \{\mathcal{E}([M_n])\} \text{ satisfies } \mathcal{E}_\infty \subset \mathcal{E}([M]),$$

and strict lower-semi-continuity means there exists $M_n \to M$ for which the final containment is proper. As remarked, when $\dim_{\mathbb{C}}(\text{Teich}(S)) = 1$, $\mathcal{E}$ is a homeomorphism [**Min3**].

Proof: We first find a point of discontinuity for $\mathcal{E}$ (to prove strict lower-semi-continuity). Since $\dim_{\mathbb{C}}(\text{Teich}(S)) > 1$ we can find a pair of distinct isotopy classes γ and δ in $\mathcal{S}$ with $i(\gamma, \delta) = 0$. Let $\mathcal{P} \subset \mathcal{S}$ be a maximal partition containing δ and γ. Adjust the Fenchel-Nielsen coordinates of $X \in \text{Teich}(S)$ along $\mathcal{P}$ so that $X_{m,n} \in \text{Teich}(S)$ has Fenchel-Nielsen coordinates

$$\text{length}_\delta(X_{m,n}) = 1/m \quad \text{and} \quad \text{length}_\gamma(X_{m,n}) = 1/n$$

and all other coordinates equal to those of X. Then, as above, the sequence $\{Q(X_{m,n}, Y)\}_{m=1}^\infty$ converges to a limit M_n for which $\mathcal{E}(M_n) = \gamma$. Likewise, the sequence $\{M_n\}_{n=1}^\infty$ converges to a limit M such that $\mathcal{E}(M) = \delta \sqcup \gamma$.

Just as a weakly convergent sequence of measures with constant support cannot converge to a measure with larger support, there is no sequence of transverse measures (weights) on the simple closed curve γ that converges in $\mathcal{ML}(S)$ to a transverse measure on $\gamma \sqcup \delta$. Hence the quasi-isometry class of M is a point of discontinuity of $\mathcal{E}$ as a map to $\mathcal{EL}(S)$ with the quotient topology.

To see that the map $\mathcal{E}$ is lower-semi-continuous in the sense of line 4.1, note that for any convergent sequence $M_n \to M$ in ∂B_Y, and any convergent sequence of measured laminations $\mu_n \to \mu$ with $|\mu_n| = \mathcal{E}(M_n)$, we have

$$\underline{\mathrm{length}}_{M_n}(\mu_n) = 0$$

for each n. Continuity of $\underline{\mathrm{length}}$ implies that $\underline{\mathrm{length}}_M(\mu) = 0$, and we conclude

$$|\mu| \subset \mathcal{E}(M).$$

■

Spinning maximal cusps. We briefly give another example of discontinuity of $\mathcal{E}$ in the quotient topologies. We do this to motivate a new topology on the range, which we introduce in the next section.

Let $C \subset \mathcal{S}$ be a maximal partition. Then the *maximal cusp* $M(C) \in \partial B_Y$ is the unique point for which α is parabolic for each $\alpha \in C$. It is determined up to isometry by the collection C (see, e.g. [**Bers2**], [**Mc2**]).

As above, assume $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) > 1$, let C_0 be a maximal partition for S, and let $\gamma \sqcup \delta \subset \mathcal{S}$ be isotopy classes of disjoint simple closed curves so that $i(\alpha, \gamma)$ and $i(\alpha, \delta)$ are non-zero for each $\alpha \in C_0$.

Let τ_γ and τ_δ be Dehn-twists about γ and δ respectively, and let

$$C_n = \tau_\gamma^{n^2} \circ \tau_\delta^n(C_0),$$

where $n \in \mathbb{N}$. Consider any limit M of the sequence of maximal cusps $\{M(C_n)\}_{n=0}^\infty$.

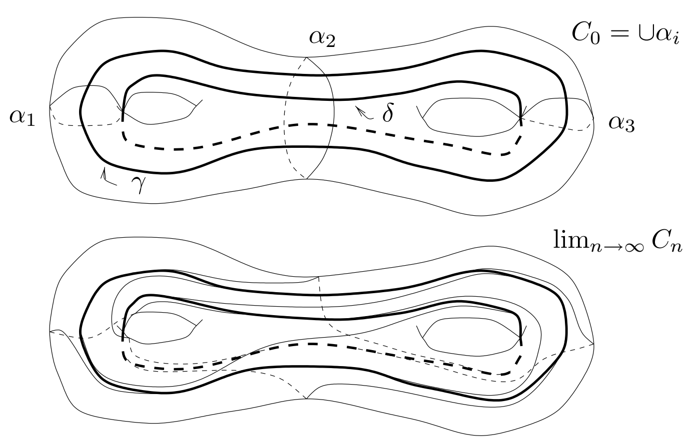

Figure 2. Spinning maximal cusps. The Hausdorff limit of $C_n = \tau_\gamma^{n^2} \circ \tau_\delta^n(C_0)$ contains both γ and δ as measurable sub-laminations.

Notice that

1. Any sequence $\mu_n \in \mathcal{ML}(S)$ of measures (weights) on C_n has projective classes $[\mu_n] \in \mathcal{PL}(S)$ converging to $[1 \cdot \gamma]$. Thus theorem 4.1 guarantees only that γ is parabolic in M.

2. One expects that *both* classes γ and δ are parabolic in M.[3]

The topology on $\mathcal{PL}(S)$ is insensitive to all but the maximal growth rate of transverse measure. Our goal in the next section will be to formulate a topology on $\mathcal{EL}(S)$ called the *end-invariant topology* that is sensitive to different orders of convergence. Proving continuity of $\mathcal{E}$ in the end-invariant topology, we capture more geometric information about general limits M.

5. Continuity in the end-invariant topology

Definition 5.1. *The* end-invariant topology *on $\mathcal{EL}(S)$ is the topology of convergence for which $\nu_n \to \nu$ if for any Hausdorff limit λ_H of any subsequence ν_{n_j}, the maximal measurable sub-lamination $\eta \subset \lambda_H$ is a sub-lamination of ν.*

Continuity in the end-invariant topology relies on uniform estimates for the shapes of *train tracks* in 3-manifolds.

Definition 5.2. *A train track τ in a hyperbolic surface $X \in$ Teich(S) is an embedded 1-complex in X whose edges (branches) are C^1 arcs meeting at vertices (switches) so that each switch v has a neighborhood $U \subset X$ for which $\tau \cap U$ is a collection of C^1 arcs passing through with a common tangent line at v. We require in addition that the double of each component of $X - \tau$ along the interiors of the branches in its boundary has negative Euler characteristic.*

A *train-path* r is a monotone C^1 immersion $r\colon \mathbb{R} \to X$ (r is "bi-infinite") or $r\colon S^1 \to X$ (r is "closed") with image in τ. A train track τ on X *carries* a geodesic lamination λ if there is a C^1 map $p\colon X \to X$ that is homotopic to the identity and non-singular on the tangent spaces to the leaves of λ so that p sends each leaf of λ to a train-path for τ. We say τ *minimally carries* λ if for each branch b of τ, there is a train-path corresponding to a leaf of λ that traverses b.

A train track τ^* in a marked hyperbolic manifold $(f\colon S \to M) \in AH(S)$ is a train track τ on a hyperbolic surface $(h\colon S \to X) \in$ Teich(S), together with a marking-preserving smooth map $g\colon X \to M$ so that $g(\tau) = \tau^*$. The surface X serves to mark the train track τ^* with homotopy information: we say τ^* carries λ if τ does.

To make a train-track τ carry more laminations, we may *enlarge* τ by adding branches. For our purposes, we enlarge τ by adding branches b each endpoint of which either terminates in a switch of τ or attaches to a simple closed curve component of τ.

Finally, a train track τ in X (or in M) is *ϵ-nearly-straight* if each train path r is C^2 with geodesic curvature less than ϵ. An important property of nearly-straight train tracks is the following: for any $\epsilon_0 \in (0,1)$ there is a "tracking constant" $C_{\mathrm{tr}} > 1$ so that for any $\epsilon \in (0, \epsilon_0)$ if τ is an ϵ-nearly-straight train track in X (resp. M), any train path r lifts to an embedding $\widetilde{r}\colon \mathbb{R} \to \mathbf{P}\mathbb{H}^2$ into the projective unit tangent bundle $\mathbf{P}\mathbb{H}^2$ of $\mathbb{H}^2$ (resp. $\mathbf{P}\mathbb{H}^3$) that is smoothly homotopic to a complete geodesic by an isotopy that moves each point a distance less than $C_{\mathrm{tr}}\epsilon$. Assume $\epsilon_0 = 1/2$ and let C_{tr} be the corresponding tracking constant.

When a closed train-path on an ϵ-nearly-straight train track is straightened to its geodesic representative, its arc-length does not decrease too much: there is a

[3] This follows, for example, from the techniques of [**KT**] and [**Br2**] and a study of the *geometric limit* of $M(C_n)$; we develop a point of view more closely aligned with the present techniques.

continuous *contraction bound* $K: [0,1) \to [1,\infty)$ with $K(\epsilon) \to 1$ as $\epsilon \to 0$ so that any arc $\alpha \in \mathbb{H}^n$ of geodesic curvature less than ϵ satisfies

$$\ell(\alpha^*) \geq \frac{1}{K(\epsilon)}\ell(\alpha) \tag{5.2}$$

where α^* is the geodesic representative of α *rel*-endpoints (see [**Br1**, §4] or [**Min1**] for more on nearly-straight train tracks)

We employ these ideas to prove the following:

Theorem 5.3. *The mapping $\mathcal{E}$ is a continuous surjection from the quotient topology on $\partial B_Y/\text{qi}$ to $\mathcal{EL}(S)$ with the end-invariant topology.*

Proof: We have shown surjectivity in theorem 3.2. It remains to show continuity in the end invariant topology.

Let $M_n \to M$ in ∂B_Y. After passing to a subsequence, let $\mathcal{E}(M_n) = \mathcal{E}_n$ tend to λ_H in the Hausdorff topology. For each n, let $P_{j,n} \subset \mathcal{S}$ be as constructed in the proof of theorem 3.2 so that $P_{j,n} \to \mathcal{E}_n$ in the Hausdorff topology as $j \to \infty$.

Arguing as in the proof of lemma 3.3, theorem 2.1 implies that given any compact set $K \in M_n$, there is a J so that for all $j > J$ no curve in $P_{j,n}$ has a geodesic representative intersecting K.

Let ν be any connected, measurable sub-lamination of λ_H. Suppose that ν is realizable in M by a pleated surface $g: X \to M$. Let $K \subset M$ be a compact set containing the radius 1 neighborhood $\mathcal{N}_1(g(\nu))$ of $g(\nu)$, the locally-isometric image of the geodesics in ν under g. By algebraic convergence, there are smooth, marking-preserving homotopy equivalences $q_n: M \to M_n$ that tend C^∞ to a local isometry on K. It follows that for any $\delta > 0$, each geodesic leaf $l \subset \nu$ has image $q_n(g(l))$ with geodesic curvature less than δ for $n \gg 0$.

Therefore we may diagonalize as follows: there is a sequence $j_n \to \infty$ so that $P_{j_n,n} = P_n$ converges to λ_H in the Hausdorff topology, and so that no curve in P_n has geodesic representative intersecting the compact sets $q_n(K)$ for $n \gg 0$.

After passing to a further subsequence, there are curves $c_n \in P_n$ that converge in the Hausdorff topology to a lamination λ' so that $\nu \subset \lambda'$. Applying the construction of nearly-straight train tracks in [**Br1**, Lem. 5.2, Cor. 5.3], there is a uniform C depending only on S and the injectivity radius along the image $g(\nu)$ of ν in M for which the following holds: for any $\epsilon > 0$

1. there exists an ϵ-nearly-straight train track $\tau \subset M$ carrying ν, and
2. τ admits an enlargement τ_n that minimally carries c_n with a $C\epsilon$-nearly-straight realization τ_n^* in M_n for $n \gg 0$.

Choosing ϵ and δ sufficiently small, then, for $n \gg 0$, both the image $q_n(g(\nu))$ and the train track τ_n^* lie close to the realization of ν in M_n and hence close to each other: precisely, $q_n(g(\nu))$ lies within $C_{\text{tr}}(C\epsilon + \delta)$ of τ_n^*, since τ_n^* carries ν. As τ_n^* also carries c_n, and τ_n^* is nearly-straight, c_n is realizable in M_n with geodesic representative c_n^*. Indeed, c_n^* lies within $C_{\text{tr}}C\epsilon$ of τ_n^* and thus within $C_{\text{tr}}(2C\epsilon+\delta)$ of $q_n(g(\nu))$. We have a contradiction, since either c_n is non-realizable, or its geodesic representative c_n^* lies outside $q_n(K)$ for all n sufficiently large.

The contradiction implies that ν is not realizable in M, and hence $\nu \subset \mathcal{E}(M)$.

∎

6. Convergence in Bers' compactification

The above methods bear on the question of how the divergent surfaces $X_n \in \text{Teich}(S)$ for which $Q(X_n, Y) \to M \in \partial B_Y$ and the quotient manifolds $M_n = Q(X_n, Y)$ determine the end invariant $\mathcal{E}(M)$ of their limit in Bers' boundary.

A direct consequence of theorem 4.1 is the following:

THEOREM 6.1. *Let $X_n \to [\mu]$ in Thurston's boundary $\mathcal{PL}(S)$ for* Teich(S). *Then for any limit $M \in \partial B_Y$ of $\{Q(X_n, Y)\}$, we have $|\mu| \subset \mathcal{E}(M)$.*

Proof: In [**Th5**], Thurston constructs measured laminations μ_n so that $\mu_n \to \mu$ in $\mathcal{ML}(S)$, and $\text{length}_{X_n}(\mu_n) \to 0$. The theorem follows from an application of theorem 4.1.

■

As with maximal cusps, however, the support $|\mu|$ of the limit lamination $[\mu] \in \mathcal{PL}(S)$ is often a small piece of $\mathcal{E}(M)$. We now formulate a construction to obtain partitions $\Pi(M_n)$ of S using the limiting geometry of M_n so that $\Pi(M_n)$ converge to $\mathcal{E}(M)$ in the end-invariant topology. We remark that various such constructions are possible, requiring various levels of detail. We present a simple one.

Constructing partitions. By a theorem of Bers (see [**Bus**, Thm. 5.2.6]), there is a uniform constant $B > 0$ depending only on S so that any given $X \in \text{Teich}(S)$ admits a maximal partition Π all of whose elements γ satisfy

$$\text{length}_X(\gamma) < B.$$

Consider a sequence $M_n = Q(X_n, Y)$ converging to $M \in \partial B_Y$, and consider the set $\mathcal{B}_n \subset \mathcal{S}$ consisting of curves of length less than B on X_n. For each n, let β_n^1 denote an element of $\mathcal{B}_n$ that minimizes the ratio

$$\frac{\text{length}_{M_n}(\beta)}{\text{length}_Y(\beta)}$$

over all elements $\beta \in \mathcal{B}_n$. Continuing inductively, let β_n^k be an element of

$$\mathcal{B}_n \cap \mathcal{S}(S - \beta_n^1 \sqcup \ldots \sqcup \beta_n^{k-1})$$

that minimizes the above ratio.

Let k_0 denote the maximal k for which the ratio

$$\frac{\text{length}_{M_n}(\beta_n^k)}{\text{length}_Y(\beta_n^k)} \to 0,$$

and let

$$\Pi(M_n) = \beta_n^1 \sqcup \ldots \sqcup \beta_n^{k_0}.$$

Then we have the following.

THEOREM 6.2. *Let $X_n \to \infty$ in* Teich(S) *determine quasi-Fuchsian manifolds $M_n = Q(X_n, Y) \to M$ in ∂B_Y. Then the partitions $\Pi(M_n)$ converge to $\mathcal{E}(M)$ in the end-invariant topology.*

Proof: Consider a Hausdorff limit λ_H of $\Pi(M_n)$. If $\alpha \in \mathcal{S}$ is an isolated simple closed curve in λ_H, then α lies in infinitely many $\Pi(M_n)$ so we have

$$\inf_n \{\text{length}_{M_n}(\alpha)\} = 0.$$

Hence $\alpha \subset \mathcal{E}(M)$, by theorem 2.3.

For any other measurable sublamination $\nu \subset \lambda_H$ there is a sequence $c_n \in \Pi(M_n)$ so that $\text{length}_Y(c_n) \to \infty$ and ν lies in the Hausdorff limit of c_n after passing to a subsequence. Assume ν is realizable in M. As in the proof of theorem 5.3, there is an ϵ-nearly-straight train track $\tau \subset M$ carrying ν, and a uniform $C > 1$ so that τ admits enlargements τ_n minimally carrying c_n with $C\epsilon$-nearly-straight realizations τ_n^* in M_n, for $n \gg 0$.

Given a branch b of τ_n, let $m_b(c_n)$ be the weight c_n assigns to b; i.e. the number of times c_n traverses b. Then by [**Br1**, Cor. 5.3] given any $b \in \tau$, the weight $m_b(c_n)$ grows without bound. Since the total length $\ell_{\tau_n^*}(c_n)$ of the train-path homotopic to c_n on τ_n^* satisfies

$$\text{length}_{M_n}(c_n) \geq \frac{1}{K(C\epsilon)} \ell_{\tau_n^*}(c_n),$$

where $K(C\epsilon)$ is the contraction bound of equation 5.2 of §5 (see also [**Br1**, §4]), it follows that $\text{length}_{M_n}(c_n)$ diverges.

Since, however, we have

$$\text{length}_{M_n}(c_n) \leq 2\text{length}_{X_n}(c_n),$$

by [**Bers2**, Thm. 3] or [**Mc1**, Prop. 6.4], it follows that $\text{length}_{M_n}(c_n) < 2B$, contradicting the divergence of $\text{length}_{M_n}(c_n)$. Thus ν is non-realizable, and therefore ν lies in $\mathcal{E}(M)$.

∎

Convergence to the boundary in $\overline{B_Y}$. We unify these two perspectives on $\mathcal{E}(M)$ as follows. Given $M \in \partial B_Y$, the conformal boundary $\partial M - Y$ is a (possibly empty) union X of hyperbolic surfaces. Given any sequence $M_n \in \overline{B_Y}$ converging to M, let $X_n = \partial M_n - Y$. We construct partitions $\Pi(M_n)$ of X_n, exactly as above: Choose pairwise disjoint curves $\beta_n^1, \ldots, \beta_n^{k_0}$ from the set $\mathcal{B}_n \subset \mathcal{S}(X_n)$ of curves of length less than B on X_n so that each β_n^k minimizes the ratio

$$\frac{\text{length}_{M_n}(\beta)}{\text{length}_Y(\beta)}$$

over all $\beta \in \mathcal{B}_n \cap \mathcal{S}(X_n - \beta_n^1 \sqcup \ldots \sqcup \beta_n^{k-1})$ and so that we have

$$\frac{\text{length}_{M_n}(\beta_n^{k_0})}{\text{length}_Y(\beta_n^{k_0})} \to 0.$$

Then the resulting union $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ is a geodesic lamination on S.

Corollary 6.3. *The laminations $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ converge to $\mathcal{E}(M)$ in the end-invariant topology.*

Proof: Pass to a subsequence so that $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ converges to λ_H in the Hausdorff topology. Then for any connected measurable sub-lamination $\nu \subset \lambda_H$, there is a further subsequence so that ν lies either in the Hausdorff limit of the partition $\Pi(M_n)$ or the laminations $\mathcal{E}(M_n)$. It follows from theorems 2.3 and the proof of theorem 6.2 that ν lies in $\mathcal{E}(M)$.

∎

7. The failure of the Hausdorff topology to predict the end-invariant

In this section we address the questions of whether the $\mathcal{E}$ can have a continuous inverse in the end-invariant topology, and whether limiting values of $\mathcal{E}$ give a complete description of the end-invariant.

The inverse $\mathcal{E}^{-1}$ is known to be well defined on points $|\mu|$ of $\mathcal{PL}(S)/|.|$ for which $|\mu|$ is a collection of simple closed curves; each M for which $\mathcal{E}(M) = |\mu|$ is quasi-isometrically unique (M is a geometrically finite cusp).[4] In the end-invariant topology, there are abundant discontinuities of $\mathcal{E}^{-1}$ on this set arising from approximation by maximal cusps. For example, given a single simple closed curve $\gamma \in \mathcal{S}$ and an M for which $\mathcal{E}(M) = \gamma$, there are maximal cusps $M(C_n)$ converging to M by the main result of [**Mc2**]. By theorem 5.3 any Hausdorff limit of C_n has γ as its unique measurable sub-lamination. In the end-invariant topology, however, any measurable lamination λ containing γ is a limit of C_n, and when $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) > 1$ there are infinitely many such λ. In this case, then, γ is necessarily a point of discontinuity for $\mathcal{E}^{-1}$ in the end-invariant topology.

In the setting of convergent maximal cusps $M(C_n) \to M$, where $\mathcal{E}(M(C_n))$ cannot be enlarged, it is natural to ask whether the maximal measurable sub-lamination ν of any Hausdorff limit of $\{C_n\}$ gives a complete picture of the end-invariant $\mathcal{E}(M)$. If C_n converges in the Hausdorff topology to a lamination that does not relatively fill (such examples are easy to arrange), lemma 3.3 shows that at the very least one must enlarge ν to the lamination $\widehat{\nu}$ (by adding any missing curves in its implicit partition) to hope for the equality $\widehat{\nu} = \mathcal{E}(M)$.

We conclude this paper with an example that shows that adding the implicit partition for ν is not in general enough to obtain this equality: new parabolics can arise that are neither contained nor implicit in ν.

Theorem 7.1. Implicit Cusps *Let* $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) > 1$, *and let* γ *lie in* $\mathcal{S}$. *Then for any* α *in* $\mathcal{S}(S-\gamma)$, *there are maximal partitions* $C_n \to \lambda_H$ *in the Hausdorff topology and associated maximal cusps* $M(C_n) \to M$ *in* ∂B_Y *for which:*

1. γ *is the maximal measurable sub-lamination of* λ_H, *and*
2. α *lies in* $\mathcal{E}(M)$.

Proof: By the assumption that $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) > 1$, there are infinitely many α satisfying the hypotheses.

We construct the sequence of maximal partitions C_n as follows. Let $\varphi \in \mathrm{Mod}(S)$ be a mapping class so that

1. φ fixes α,
2. φ restricts to a pseudo-Anosov mapping class on the closure of the component T of $S - \alpha$ containing γ
3. φ is the identity otherwise

(see [**FLP**, Exp. 9], [**Th3**], [**Br2**]). Let $\tau_\gamma \in \mathrm{Mod}(S)$ be a Dehn twist about the curve γ. Let P_0 be a maximal partition, all of whose elements cross α. Let $\varphi^k(P_0) = P_k$. By assigning weight 1 to each element of P_k we obtain a sequence $\{[P_k]\} \subset \mathcal{PL}(S)$, that converges to a limit $[\mu_\infty]$ after passing to a subsequence.

Let $\mu^u \in \mathcal{ML}(S)$ denote the unstable lamination for the pseudo-Anosov restriction of φ to T; i.e. μ^u is the unique measured lamination for which $\varphi(\mu^u) = c\mu^u$

[4]Y. Minsky recently announced $\mathcal{E}^{-1}$ is well defined on laminations of *bounded type* [**Min4**].

with $c > 1$. Noting that

$$i(\mu^u, \varphi^k(P_0)) = i(\varphi^{-k}(\mu^u), P_0) = \frac{i(\mu^u, P_0)}{c^k},$$

it follows from continuity of $i(.,.)$ (see [**Bon1**, Prop. 4.5]) that $i(\mu^u, \mu_\infty) = 0$.

Let λ be a Hausdorff limit of a subsequence of P_k. If α separates S, then let $T' = \overline{S - T}$. Then $\varphi(\beta) = \beta$ for each $\beta \in \mathcal{S}(T')$, so $i(\beta, P_k)$ does not depend on k (and is therefore bounded). Thus, λ contains no measurable sub-lamination η for which $\eta = |\mu'|$ and $\mu' \in \mathcal{ML}(T')$.

Hence, either $[\mu_\infty] = [\mu^u]$ or α is a sub-lamination of μ_∞. We wish to avoid this possibility, so we adjust each P_k by the power $m_k \in \mathbb{Z}$ of an α-Dehn twist τ_α for which the total length of

$$P'_k = \tau_\alpha^{m_k}(P_k)$$

on Y is minimized. It follows that the curves in P'_k and α realized as geodesics on Y intersect with angle uniformly bounded away from 0.

For any $\beta \in \mathcal{S}(S - \alpha)$ we have $i(\beta, P'_k) = i(\beta, P_k)$, so the above intersection number arguments apply to P'_k: after passing to perhaps further subsequences, we have $[P'_k] \to [\mu^u]$ in $\mathcal{PL}(S)$ and P'_k converge as geodesic laminations to a Hausdorff limit λ' with maximal measurable sublamination $|\mu^u|$ (see figure 3).

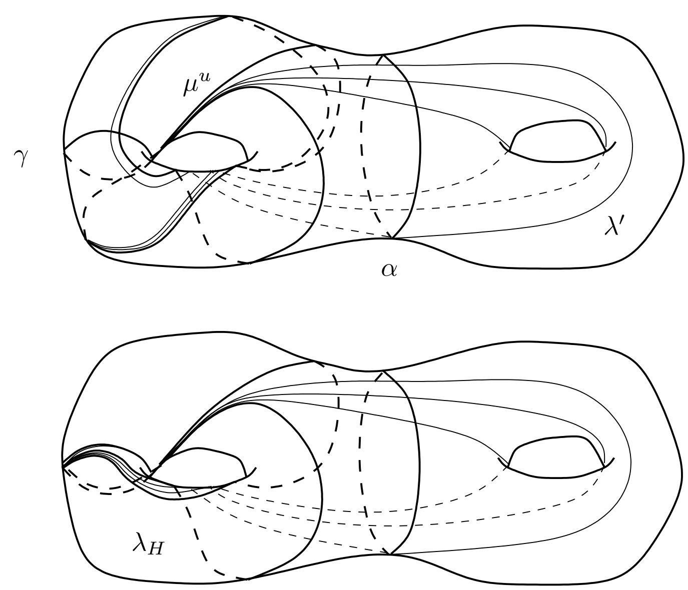

Figure 3. An implicit cusp: $\mathcal{E}(M) = \gamma \sqcup \alpha$, but α does not lie in λ_H.

Now consider the action of the Dehn twist τ_γ on λ'. Since $i(\mu^u, \gamma) > 0$ and every leaf of $|\mu^u|$ is dense in $|\mu^u|$, every leaf of $|\mu^u|$ crosses γ infinitely in each direction. Each leaf of λ' is either a leaf of $|\mu^u|$ or asymptotic to leaves of $|\mu^u|$ in each direction, so every leaf of λ' crosses γ infinitely often in each direction. The Hausdorff limit λ_H of $\{\tau_\gamma^n(\lambda')\}_{n=1}^\infty$ consists of γ together with a finite number of pairwise disjoint bi-infinite geodesics that spiral into γ from either side (figure 3).

Thus, γ is the only measurable sub-lamination of λ_H, and λ_H crosses the simple closed curve α transversely (again, as geodesics on Y). Diagonalizing, for each n we choose k_n so that $\tau_\gamma^n(P'_{k_n})$ converges to λ_H in the Hausdorff topology as $n \to \infty$. Let

$$C_n = \tau_\gamma^n(P'_{k_n}).$$

We claim that by enlarging k_n further we may guarantee that the maximal cusps $M_n = M(C_n) \in \partial B_Y$ satisfy

$$\underline{\text{length}}_{M_n}(\alpha) < \frac{1}{n}. \tag{7.3}$$

To see this, note that if we let k tend to ∞ with n fixed, the maximal cusps $\{M(\tau_\gamma^n(P'_k))\}_{k=1}^\infty$ converge up to subsequence to a limit $M_\infty(n) \in \partial B_Y$ with the property that

$$|\tau_\gamma^n(\mu^u)| \subset \mathcal{E}(M_\infty(n)).$$

Since for each n the implicit partition $\widehat{P}(|\tau_\gamma^n(\mu^u)|)$ of $|\tau_\gamma^n(\mu^u)|$ is the single simple closed curve α, lemma 3.3 guarantees that α lies in $\mathcal{E}(M_\infty(n))$. Thus, α is parabolic in $M_\infty(n)$, so the claim (inequality 7.3) follows by continuity of $\underline{\text{length}}$ (theorem 2.3).

Applying theorem 2.3 once again, we have that α is parabolic in M.

∎

A concluding remark: The reader familiar with geometric or *Gromov-Hausdorff* convergence of hyperbolic manifolds will recognize the similarity of the above example to the main example of [**KT**, §3] and others like it (cf. [**Br2**]). In the case above, the geometric limit M_G covered by M has a *degenerate end* that forces an implicit cusp at α, as well as a rank-two cusp with core-curve γ. The parabolic α lifts to M while the cusp at γ provides an obstruction to lifting the degenerate end. It would seem that a complete understanding of how values of $\mathcal{E}$ vary on Bers boundary depends, like many issues in the deformation theory, on developing a better understanding of the full spectrum of possible geometric limits of sequences $\{M_n\} \subset \partial B_Y$.

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