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THE WEIL-PETERSSON METRIC AND VOLUMES OF 3-DIMENSIONAL HYPERBOLIC CONVEX CORES

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Figure 1. The lift to $\mathbb{H}^3$ of a quasi-Fuchsian convex core boundary component.

1. Introduction

Recent insights into the combinatorial geometry of Teichmüller space have shed new light on fundamental questions in hyperbolic geometry in 2 and 3 dimensions. Paradoxically, a coarse perspective on Teichmüller space appears to refine the analogy of Teichmüller geometry with the internal geometry of hyperbolic 3-manifolds first introduced and pursued by W. Thurston.

In this paper we develop such a coarse perspective on the Weil-Petersson metric on Teichmüller space by relating it to a graph of pair-of-pants decompositions of surfaces introduced by Hatcher and Thurston. This viewpoint generates a new connection between the Weil-Petersson geometry of Teichmüller space and the geometry of the convex core of a hyperbolic 3-manifold.

For simplicity, let S be a closed oriented surface of negative Euler characteristic. A *pants decomposition* of S is a maximal collection of distinct isotopy classes of pairwise disjoint essential simple closed curves on S. We say two distinct pants

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decompositions P and P' are related by an *elementary move* if P' can be obtained from P by replacing a curve $\alpha \in P$ by a curve β intersecting α minimally (see Figure 3).

One obtains the *pants graph* $\mathbf{P}(S)$ by making each pants decomposition a vertex and joining two pants decompositions differing by an elementary move by an edge. Setting the length of each edge to 1, $\mathbf{P}(S)$ becomes a metric space. We find the graph $\mathbf{P}(S)$ provides a combinatorial model for the coarse geometry of the Weil-Petersson metric:

Theorem 1.1. *The graph $\mathbf{P}(S)$ is naturally quasi-isometric to Teichmüller space with the Weil-Petersson metric.*

The connection to hyperbolic 3-manifolds is simple to describe. By a theorem of Bers, a pair of points $(X,Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ naturally determines a quasi-Fuchsian hyperbolic 3-manifold $Q(X,Y) \cong S \times \mathbb{R}$ with X and Y in its conformal boundary at infinity. Its *convex core*, denoted $\mathrm{core}(Q(X,Y))$, is the smallest convex subset of $Q(X,Y)$ carrying its fundamental group. The convex core is itself homeomorphic to $S \times I$ and carries all the essential geometric information about the manifold $Q(X,Y)$.

Because $Q(X,Y)$ is obtained from the pair (X,Y) by an analytic process (Bers's *simultaneous uniformization*), it is a central challenge in the study of hyperbolic 3-manifolds to understand the geometry of $Q(X,Y)$ purely in terms of the geometry of X and Y. Our main theorem proves a conjecture of Thurston that the following fundamental connection exists between convex core *volume* and the Weil-Petersson distance.

Theorem 1.2. *The volume of the convex core of $Q(X,Y)$ is comparable to the Weil-Petersson distance $d_{\mathrm{WP}}(X,Y)$.*

Here, *comparability* means that two quantities are equal up to uniform additive and multiplicative error: i.e. there are constants $K_1 > 1$ and $K_2 > 0$ depending only on S so that for any $(X,Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ we have

$$\frac{d_{\mathrm{WP}}(X,Y)}{K_1} - K_2 \leq \mathrm{vol}(\mathrm{core}(Q(X,Y))) \leq K_1 d_{\mathrm{WP}}(X,Y) + K_2.$$

Throughout the paper we will use the contraction $\mathrm{vol}(X,Y) = \mathrm{vol}(\mathrm{core}(Q(X,Y)))$ and the notation $\asymp$ to denote the comparability of two quantities; then Theorem 1.2 becomes

$$d_{\mathrm{WP}}(X,Y) \asymp \mathrm{vol}(X,Y).$$

The volume of the convex core of a complete hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is directly related to the lowest eigenvalue of the Laplacian on M as well as the Hausdorff dimension of the *limit set* $\Lambda(\Gamma) \subset \widehat{\mathbb{C}}$, namely the complement of the invariant *domain of discontinuity* $\Omega(\Gamma) \subset \widehat{\mathbb{C}}$ where the action of Kleinian covering group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ for M is properly discontinuous (see Figure 2 for two examples of limit sets[1]).

As an immediate application, Theorem 1.2 implies the following new relationship between these analytic invariants and the Weil-Petersson distance. Let $\lambda_0(X,Y)$ denote the lowest eigenvalue of the Laplacian on the quasi-Fuchsian hyperbolic 3-manifold $Q(X,Y) = \mathbb{H}^3/\Gamma(X,Y)$ and let $D(X,Y)$ denote the Hausdorff dimension of the limit set of $\Gamma(X,Y)$.

[1] We have employed computer programs of Curt McMullen in our generation of Figures 1 and 2.

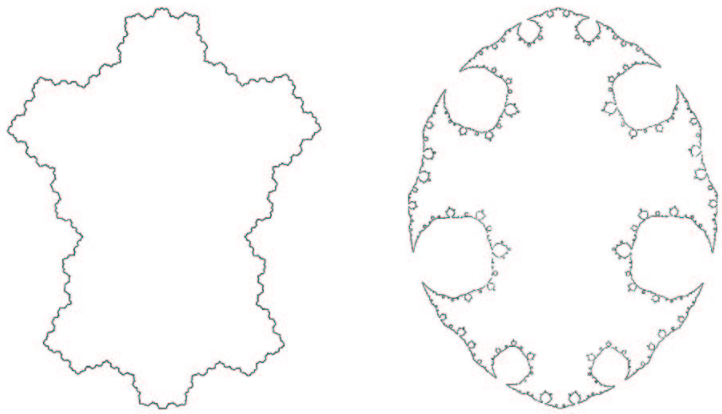

Figure 2. Limit sets for two quasi-Fuchsian groups.

Theorem 1.3. *Given S there are constants $K > 0$, C_1, C_2, C_3, and $C_4 > 1$ so that if $d_{\mathrm{WP}}(X,Y) > K$ then*

$$\frac{C_1}{d_{\mathrm{WP}}(X,Y)^2} \le \lambda_0(X,Y) \le \frac{C_2}{d_{\mathrm{WP}}(X,Y)},$$

and

$$2 - \frac{C_3}{d_{\mathrm{WP}}(X,Y)} \le D(X,Y) \le 2 - \frac{C_4}{d_{\mathrm{WP}}(X,Y)^2}.$$

Proof: The relation $\lambda_0(X,Y) = D(X,Y)(2 - D(X,Y))$ follows from a general result by D. Sullivan (see [Sul2, Thm. 2.17]), after applying Bowen's Theorem [Bow] that $D(X,Y) \ge 1$ with equality if and only if $X = Y$.

Theorem 1.2 may be rephrased to claim the existence of K, K' so that for $d_{\mathrm{WP}}(X,Y) > K$ we have

$$\frac{d_{\mathrm{WP}}(X,Y)}{K'} \le \mathrm{vol}(X,Y) \le K' d_{\mathrm{WP}}(X,Y).$$

The theorem then follows from the double inequality

$$\frac{c_1}{\mathrm{vol}(X,Y)^2} \le \lambda_0(X,Y) \le \frac{c_2}{\mathrm{vol}(X,Y)}$$

(see [BC, Main Thm.] and [Can1, Thm. A]) after collecting constants.

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The pants graph. Since Theorem 1.2 relies directly on Theorem 1.1 we detail our coarse perspective on the Weil-Petersson metric.

To describe the nature of the quasi-isometry between the graph $\mathbf{P}(S)$ and the Weil-Petersson metric, we recall that by a theorem of Bers, there is a constant $L > 0$ depending only on S so that for each $X \in \mathrm{Teich}(S)$ there is a pants decomposition P so that

$$\ell_X(\alpha) < L \quad \text{for each} \quad \alpha \in P,$$

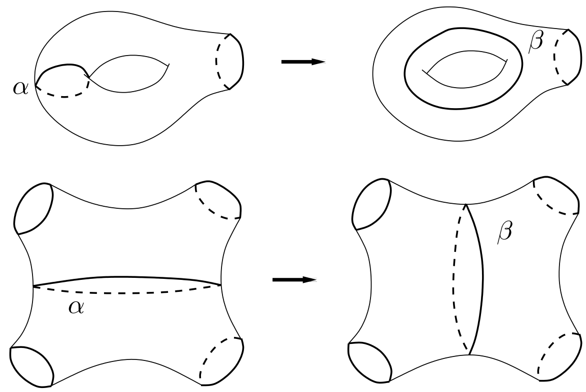

Figure 3. Elementary moves on pants decompositions.

where $\ell_X(\alpha)$ denotes the length of the geodesic representative of α in the hyperbolic metric on X.

If P is a pants decomposition, we denote by $V_\ell(P)$ the sub level set

$$V_\ell(P) = \left\{ X \mid \max_{\alpha \in P} (\ell_X(\alpha)) < \ell \right\}.$$

Then Bers's theorem guarantees that the sub level sets $V(P) = V_L(P)$ cover Teichmüller space.

Coarsely, the Weil-Petersson distance records the configuration of the sub level sets $V(P)$ with $\mathbf{P}(S)$ as its model. Indeed, if P_X and P_Y are pants decompositions for which $X \in V(P_X)$ and $Y \in V(P_Y)$, then the quasi-isometry of Theorem 1.1 arises from the comparability

$$d_{\mathbf{P}}(P_X, P_Y) \asymp d_{\mathrm{WP}}(X, Y)$$

where $d_{\mathbf{P}}(P_X, P_Y)$ is the minimal number of elementary moves required to travel from P_X to P_Y in $\mathbf{P}(S)$.

We briefly outline our proof of Theorem 1.1 and its application to Theorem 1.2.

Outline of the proof of Theorem 1.1. Let $\mathbf{P}^0(S)$ denote the vertex set of $\mathbf{P}(S)$, and let $Q \colon \mathbf{P}^0(S) \to \mathrm{Teich}(S)$ be any map so that $Q(P) \in V(P)$. Applying work of S. Wolpert and H. Masur there is a uniform constant $D > 0$ so that the Weil-Petersson diameter of $V(P)$ satisfies

$$\mathrm{diam}_{\mathrm{WP}}(V(P)) < D$$

for all $P \in \mathbf{P}^0(S)$. A simple argument shows that for any two pants decompositions P_1 and P_2 differing by an elementary move there is a single Riemann surface X on which all curves in P_i have length less than L. Thus $V(P_1) \cap V(P_2)$ is non-empty, and Q is $2D$-Lipschitz.

Given two pants decompositions P_1 and P_2 for which $V_{2L}(P_1) \cap V_{2L}(P_2) \neq \emptyset$, there is an $X \in V_{2L}(P_1) \cap V_{2L}(P_2)$: i.e. a single Riemann surface on which each curve in $P_1 \cup P_2$ has length at most $2L$. It follows that there is a uniform C so that the total intersection number satisfies

$$i(P_1, P_2) \leq C,$$

which in turn provides a uniform bound to the distance $d_{\mathbf{P}}(P_1, P_2)$ in $\mathbf{P}(S)$.

A compactness argument shows that each $X \in V(P) = V_L(P)$ lies a uniformly definite distance from $\partial V_{2L}(P)$. Thus, a unit-length Weil-Petersson geodesic can always be covered by a uniform number of sub level sets $V_{2L}(P)$. It follows that any pair of pants decompositions P and P' for which $V(P)$ and $V(P')$ contain the endpoints of a unit length Weil-Petersson geodesic, P and P' have uniformly bounded distance in $\mathbf{P}(S)$, and the theorem follows.

Outline of the proof of Theorem 1.2. The proof has two parts.

Bounding volume from below: The bound below of core volume in terms of the Weil-Petersson distance begins with an interpolation through the convex core

$$h_t \colon Z_t \to \mathrm{core}(Q(X,Y))$$

of 1-Lipschitz maps of hyperbolic surfaces. It follows that for each essential simple closed curve α we have

$$\inf_t \ell_{Z_t}(\alpha) \geq \ell_{Q(X,Y)}(\alpha).$$

The path Z_t, then, only passes through sets $V(P)$ for which each element in P has length less than L in $Q(X,Y)$. Applying recent work of Masur and Minsky, we show if a sequence $\{P_1, \ldots, P_n\}$ of pants decompositions is built from N curves and makes bounded jumps, i.e.

$$d_{\mathbf{P}}(P_j, P_{j+1}) < k,$$

then its ends satisfy the bound $d_{\mathbf{P}}(P_1, P_n) < K_0 N$, where K_0 depends only on k and S.

The Margulis lemma forces closed geodesics with length less than L in $Q(X,Y)$ that represent different isotopy classes to be uniformly equidistributed through the convex core. Since each such representative makes a definite contribution to core volume, the lower bound follows.

Bounding volume from above. Given pants decompositions P_X and P_Y so that $X \in V(P_X)$ and $Y \in V(P_Y)$, and a geodesic $G \subset \mathbf{P}(S)$ joining P_X to P_Y, we consider the closed geodesics

$$\mathrm{spin}(G) = \{\alpha^* \mid \alpha \in P \text{ for } P \in G\}$$

where α^* denotes the geodesic representative of α in $Q(X,Y)$. We build a straight triangulation $\mathcal{T}$ of all but a uniformly bounded volume portion of $\mathrm{core}(Q(X,Y))$ so that vertices of $\mathcal{T}$ lie on $\alpha^* \in \mathrm{spin}(G)$, the so-called *spinning geodesics*.

Our triangulation has the property that all but constant times $d_{\mathbf{P}}(P_X, P_Y)$ of the tetrahedra in $\mathcal{T}$ have at least one edge in a spinning geodesic α^*. We then use a spinning trick: by homotoping the vertices around the geodesics in $\mathrm{spin}(G)$ keeping the triangulation straight, all tetrahedra with an edge in any α^* can be made to have arbitrarily small volume.

Since there is an *a priori* bound to the volume of a tetrahedron in $\mathbb{H}^3$, the remaining tetrahedra have uniformly bounded volume. The theorem then follows from the comparability $d_{\mathrm{WP}}(X,Y) \asymp d_{\mathbf{P}}(P_X, P_Y)$.

Geometrically finite hyperbolic 3-manifolds. We remark that simple generalizations of these techniques may be employed to obtain estimates for core volume of hyperbolic 3-manifolds that are not quasi-Fuchsian once the appropriate version of Weil-Petersson distance is defined. For example, given a hyperbolic 3-manifold M_ψ that fibers over the circle with monodromy ψ, the volume of M_ψ is comparable to the Weil-Petersson translation distance of ψ (with constants depending only on the topology of the fiber). We take up these generalizations in [Br3].

Algebraic and geometric limits. As an application of Theorem 1.2, boundedness of the Weil-Petersson distance $d_{\mathrm{WP}}(X_k, Y_k)$ for sequences predicts the geometric finiteness of the geometric limit of $Q(X_k, Y_k)$.

The space $QF(S)$ of all quasi-Fuchsian hyperbolic 3-manifolds lies in the space $AH(S)$ of all complete hyperbolic 3-manifolds M marked by homotopy equivalences $(h\colon S \to M)$ so that h_* sends peripheral elements of $\pi_1(S)$ to parabolic elements of $\pi_1(M)$. The space $AH(S)$ carries the *algebraic topology* or the compact-open topology on the induced representations $h_*\colon \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ up to conjugacy.

In an algebraically convergent sequence $\{(h_k\colon S \to M_k)\}$ in $AH(S)$, normalizing the induced representations $\rho_k = (h_k)_*$ to converge on generators one may always extract a subsequence so that the groups $\rho_k(\pi_1(S)) = \Gamma_k$ converge in the Gromov-Hausdorff topology on discrete subgroups of $\mathrm{Isom}^+(\mathbb{H}^3)$, or *geometrically*, to a limit Γ_G. A central issue in the deformation theory of hyperbolic 3-manifolds is to understand the *geometric limit* $N_G = \mathbb{H}^3/\Gamma_G$.

Applying Theorem 1.2, we obtain the following criterion:

Theorem 1.4. *Let $Q(X_k, Y_k) \to Q_\infty$ be an algebraically convergent sequence in $AH(S)$ with geometric limit N_G. Then N_G is geometrically finite if and only if there is a $K > 0$ for which*

$$d_{\mathrm{WP}}(X_k, Y_k) < K$$

for all k.

Note that geometric finiteness of N_G implies geometric finiteness of Q_∞ but *not* conversely.

History and references. The fundamental properties of the Weil-Petersson metric we use are discussed in [Wol1], [Wol3], [Wol4] and [Mas]. The pants graph is the 1-skeleton of the *pants complex*, introduced in [HT] (see also [HLS]) which is there proven to be connected. The relation of the pants graph to the Weil-Petersson metric is similar in spirit to the relative hyperbolicity theorem for Teichmüller space of [MM1] where the (related) *complex of curves* is shown to be quasi-isometric to the *electric* Teichmüller space, and to be Gromov-hyperbolic (the pants complex and the Weil-Petersson metric are *not* in general Gromov-hyperbolic [BF]). For more on quasi-Fuchsian manifolds and their algebraic and geometric limits, see [Th1], [Brs1], [Mc2], [Mc1], [Br2], and [Ot].

Plan of the paper. After discussing the fundamental work of S. Wolpert and H. Masur on the Weil-Petersson metric that will serve as our jumping off point in section 2, we prove the comparability of Weil-Petersson distance and pants distance (Theorem 1.1) in section 3. We then establish the lower bound on $\mathrm{vol}(X, Y)$ in terms of the distance $d_{\mathbf{P}}(P_X, P_Y)$ in section 4. Section 5 applies the combinatorics of pants decompositions along a geodesic $G \subset \mathbf{P}(S)$ joining P_X to P_Y to bound volume from above in terms of pants distance. Theorem 1.2 then follows from the comparability of Theorem 1.1. We conclude with applications to the study of geometric limits, proving Theorem 1.4.

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2. The extended Weil-Petersson metric

Let S be a compact oriented surface of negative Euler characteristic. We allow S to have boundary and let $\mathrm{int}(S)$ denote its interior. Let $\mathcal{S}$ denote the set of isotopy classes of essential, non-peripheral, simple closed curves on S.

A *pants decomposition* $P \subset \mathcal{S}$ is a maximal collection of isotopy classes with pairwise disjoint representatives on S. The usual geometric intersection number $i(\alpha, \beta)$ of a pair $(\alpha, \beta) \in \mathcal{S} \times \mathcal{S}$ generalizes to a total intersection number $i(P, P')$ of pants decompositions by summing the geometric intersections of their components.

The *Teichmüller space* $\mathrm{Teich}(S)$ of S parameterizes finite area hyperbolic structures on S up to isotopy. Points in $\mathrm{Teich}(S)$ are pairs (f, X) where X is a finite area hyperbolic surface X equipped with a homeomorphism $f\colon \mathrm{int}(S) \to X$, up to the equivalence $(f, X) \sim (g, Y)$ if there is an isometry $\phi\colon X \to Y$ for which $\phi \circ f \simeq g$. A pants decomposition $P = \alpha_1 \cup \ldots \cup \alpha_{|P|}$ determines *Fenchel-Nielsen coordinates*

$$(\ell_X(\alpha_1), \ldots, \ell_X(\alpha_{|P|}), \theta_X(\alpha_1), \ldots, \theta_X(\alpha_{|P|})) \in \mathbb{R}_+^{|P|} \times \mathbb{R}^{|P|}$$

for each $X \in \mathrm{Teich}(S)$, indicating X is assembled from hyperbolic pairs of pants with boundary lengths prescribed by $\ell_X(\alpha_i)$ glued together twisted by $\theta_X(\alpha_i)$. (For more on Teichmüller space and Fenchel-Nielsen coordinates see [IT] or [Gard]).

The Weil-Petersson metric. Each $X \in \mathrm{Teich}(S)$ is naturally a complex 1-manifold via its uniformization $X = \mathbb{H}^2/\Gamma$ as the quotient of the upper half plane by a Fuchsian group. The Teichmüller space has a complex manifold structure of dimension $3g - 3 + n$ where S has genus g and n boundary components.

The space of holomorphic quadratic differentials $Q(X)$ on $X \in \mathrm{Teich}(S)$ (holomorphic forms of type $\phi(z)dz^2$ on X) is naturally the cotangent space $T_X^*\mathrm{Teich}(S)$ to $\mathrm{Teich}(S)$ at X. The *Weil-Petersson metric* on $\mathrm{Teich}(S)$ unifies the hyperbolic and holomorphic perspectives on X: it arises from the L^2 inner product on $Q(X)$, namely

$$\langle \varphi, \psi \rangle_{\mathrm{WP}} = \int_X \frac{\varphi \bar{\psi}}{\rho^2}$$

where $\rho(z)|dz|$ is the hyperbolic metric on X, by the usual pairing

$$(\mu, \varphi)_X = \int_X \mu\varphi$$

between $T_X\mathrm{Teich}(S)$ and $T_X^*\mathrm{Teich}(S)$ (see, e.g. [Wol3, Sec. 1]). In what follows, we will be interested only in the Riemannian part g_{WP} of the Weil-Petersson metric, and its associated distance function $d_{\mathrm{WP}}(.,.)$ on $\mathrm{Teich}(S)$.

The Weil-Petersson metric has negative sectional curvature [Tro] [Wol2], and the *modular group* $\mathrm{Mod}(S)$ (the group of isotopy classes of orientation preserving homeomorphisms of S) acts by isometries of g_{WP}. Thus, g_{WP} descends to a metric on the Moduli space $\mathcal{M}(S) = \mathrm{Teich}(S)/\mathrm{Mod}(S)$.

Work of S. Wolpert shows two important properties of the Weil-Petersson metric we will use:

WPI *The Weil-Petersson metric is not complete:* "pinching geodesics" in the Teichmüller metric (which leave every compact set of $\mathrm{Teich}(S)$) have finite Weil-Petersson length [Wol1].

WPII *The Weil-Petersson metric is geodesically convex:* in fact, for $\alpha \in \mathcal{S}$ the length function $\ell_{(.)}(\alpha)$ is strictly convex along Weil-Petersson geodesics [Wol3].

The augmented Teichmüller space. In [Mas], H. Masur shows the Weil-Petersson metric extends to the *augmented Teichmüller space* $\overline{\mathrm{Teich}(S)}$ obtained by adding boundary Teichmüller spaces consisting of marked *noded* Riemann surfaces, which we now describe (see [Brs3] for a detailed discussion).

A *Riemann surface with nodes* W is a connected complex space so that each point $p \in W$ has a neighborhood isomorphic to $\{z \in \mathbb{C} \mid |z| < 1\}$ or isomorphic to $\{(z, w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1, \text{ and } zw = 0\}$ by an isomorphism sending p to $(0, 0) \in \mathbb{C}^2$. In the latter case, p is called a *node* of X. The complement of the nodes is a union of Riemann surfaces called the *pieces* of W. We say W is *hyperbolic* if each piece of W admits a complete finite-area hyperbolic structure.

Given curves $\nu_1, \ldots, \nu_j$ in a pants decomposition $P = \{\alpha_1, \ldots, \alpha_{|P|}\}$ of S, a *marked noded hyperbolic surface pinched along* $\nu_1, \ldots, \nu_j$ is a noded hyperbolic Riemann surface W together with a continuous map

$$f : \mathrm{int}(S) \to W$$

so that $f|_{S-\nu_1\cup\ldots\cup\nu_j}$ is a homeomorphism on to the union of the pieces of W. Let $S - \mathcal{N}(\nu_1) \cup \ldots \cup \mathcal{N}(\nu_j) = S_1 \cup \ldots \cup S_k$ where $\mathcal{N}(\nu_i)$ are pairwise disjoint open collars about each ν_i. Then the pair (f, W) determines a point

$$\mathrm{Teich}(S_1) \times \ldots \times \mathrm{Teich}(S_k)$$

in the product Teichmüller space by taking the restriction of f to each component of $\mathrm{int}(S) - \nu_1 \cup \ldots \cup \nu_j$ as a marking on each piece of W. A marked piece of $W_l \in \mathrm{Teich}(S_l)$, $1 \le l \le k$, has Fenchel-Nielsen coordinates with respect to the elements of the pants decomposition P that lie in S_l.

Two marked hyperbolic noded surfaces (f_1, W_1) and (f_2, W_2) are equivalent, if there is continuous map $\phi\colon W_1 \to W_2$ that is isometric on each piece of W_1 for which $\phi \circ f_1 = f_2$ after precomposition with an isotopy of S.

The *augmented Teichmüller space* $\overline{\mathrm{Teich}(S)}$ is obtained by adjoining equivalence classes of marked noded hyperbolic surfaces to $\mathrm{Teich}(S)$. The topology on $\overline{\mathrm{Teich}(S)}$ is given as follows. Given a pants decomposition P, and a point $W \in \overline{\mathrm{Teich}(S)}$ with curves $\nu_1 \cup \ldots \cup \nu_j$ in P pinched to nodes, we extend the Fenchel-Nielsen coordinates to W by defining the coordinates $\ell_W(\nu_i) = 0$. Then a neighborhood of W in $\overline{\mathrm{Teich}(S)}$ consists of (possibly noded) hyperbolic Riemann surfaces X whose length coordinates $\ell_X(\alpha_p)$ are close to those of W for $p = 1, \ldots, 3g - 3 + n$, and whose twist coordinates $\theta_X(\alpha_p)$ are close to those of X for each p such that $\alpha_p \neq \nu_i$. (see [IT, App. B]).

The Weil-Petersson metric extends to the augmented Teichmüller space as its completion (see [Mas]), giving a $\mathrm{Mod}(S)$ invariant metric on $\overline{\mathrm{Teich}(S)}$. The quotient

$$\overline{\mathrm{Teich}(S)}/\mathrm{Mod}(S) = \overline{\mathcal{M}(S)},$$

the familiar Mayer-Mumford-Deligne compactification of the moduli space (see [Brs3]), inherits a complete extension of the Weil-Petersson metric on $\mathcal{M}(S)$. We denote the corresponding distance by

$$d_{\overline{\mathrm{WP}}} \colon \overline{\mathrm{Teich}(S)} \times \overline{\mathrm{Teich}(S)} \to \mathbb{R}_{\ge 0}.$$

Evidently, the failure of completeness of the Weil-Petersson metric occurs at limits of *pinching sequences* X_t for which the length coordinates $\ell_{X_t}(\nu_i)$ tend to zero for some collection of curves in a pants decomposition P.

Given a pants decomposition P and a collection $\alpha_1, \ldots, \alpha_k$ of curves in P, the minimal distance from a point $X \in \mathrm{Teich}(S)$ to a noded Riemann surface Z with nodes along $\alpha_1, \ldots, \alpha_k$ is estimated in terms of the geodesic length sum

$$\ell = \ell_X(\alpha_1) + \ldots + \ell_X(\alpha_k)$$

of the lengths of α_i on X by

$$d_{\overline{\mathrm{WP}}}(X, Z) = \sqrt{2\pi\ell} + O(\ell^2) \tag{2.1}$$

(see [Wol5, Cor. 21]).

Remark: This estimate is a recent improvement of similar estimates originally obtained in [Wol4, Ex. 4.3] and cited in earlier versions of this manuscript.

Sub level sets. We recall the following theorem of Bers.

Theorem 2.1 (Bers)**.** *There is a constant $L > 0$ depending only on S such that for any $X \in \mathrm{Teich}(S)$ there is a pants decomposition P such that $\ell_X(\alpha) < L$ for each $\alpha \in P$.*

We call this L the *Bers constant* for S.

Given a pants decomposition P, and a positive real number $\ell \in \mathbb{R}_+$, we consider the sub level set

$$V_\ell(P) = \left\{ X \in \mathrm{Teich}(S) \mid \max_{\alpha \in P} \{\ell_X(\alpha)\} < \ell \right\}.$$

Then by Bers's theorem, the union of the sets $V_L(P)$ over all pants decompositions gives an open cover of $\mathrm{Teich}(S)$. Because L depends only on S, we abbreviate

$$V(P) = V_L(P).$$

Then we have the following:

Proposition 2.2. *The sub level sets $V(P)$ have the following properties.*

1. *Each $V(P)$ is convex in the Weil-Petersson metric, and*
2. *there is a constant $D > 0$, depending only on S, for which the Weil-Petersson diameter* $\mathrm{diam}_{\mathrm{WP}}(V(P)) < D$.

Proof: Geodesic convexity of $V(P)$ follows immediately from **WPII**, the convexity of the geodesic length functions $\ell_X(.)(\alpha)$ for each $\alpha \in P$.

To see each $V(P)$ has bounded Weil-Petersson diameter, let W_P be the (unique) maximally noded Riemann surface where each curve in P is pinched. By equation (2.1) there is a constant $C(L)$ so that for each $X \in V(P)$ we have

$$d_{\overline{\mathrm{WP}}}(X, W_P) < C(L).$$

By the triangle inequality for $d_{\overline{\mathrm{WP}}}$, if X and Y lie in $V(P)$ then the distance $d_{\overline{\mathrm{WP}}}(X, Y)$ is bounded by $2C(L)$. By geodesic convexity of $V(P)$ the geodesic joining X to Y lies in $V(P)$, so we have the bound

$$d_{\mathrm{WP}}(X, Y) < 2C(L)$$

on $d_{\mathrm{WP}}(X, Y)$ which we set equal to D.

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3. A combinatorial Weil-Petersson distance

In this section, we relate the coarse geometry of the Weil-Petersson metric to the *pants graph* $\mathbf{P}(S)$ defined in the introduction. We do this by exhibiting a *quasi-isometry* between the two spaces with their respective distances.

Definition 3.1. *Given $k_1 > 1$ and $k_2 > 0$, a map $f\colon (X,d) \to (Y,d')$ of metric spaces is a (k_1,k_2)-*quasi-isometric embedding *if for each pair of points x and y in X we have*

$$\frac{d(x,y)}{k_1} - k_2 \le d'(f(x),f(y)) \le k_1 d(x,y) + k_2.$$

The spaces (X,d) and (Y,d') are *quasi-isometric* if for some $k_1 > 1$ and $k_2 > 0$ there are (k_1,k_2)-quasi-isometric embeddings from (X,d) to (Y,d') and from (Y,d') to (X,d). In practice, it suffices to exhibit a *quasi-isometry* from (X,d) to (Y,d'), namely, a quasi-isometric embedding with uniformly dense image. Given such a quasi-isometry from (X,d) to (Y,d'), a quasi-isometric embedding from (Y,d') to (X,d) is readily constructed, so the spaces are quasi-isometric.

Let

$$Q\colon \mathbf{P}^0(S) \to \mathrm{Teich}(S)$$

be any embedding of the vertices $\mathbf{P}^0(S)$ of $\mathbf{P}(S)$ into $\mathrm{Teich}(S)$ so that $Q(P)$ lies in $V(P)$. The main theorem of this section is the following:

Theorem 3.2. *The map Q is a quasi-isometry of the 0-skeleton $\mathbf{P}^0(S)$ of $\mathbf{P}(S)$ with $\mathrm{Teich}(S)$ with its Weil-Petersson distance.*

Proof: By the uniform bound

$$\mathrm{diam}_{\mathrm{WP}}(V(P)) < D$$

on the diameter of $V(P)$, the image $Q(\mathbf{P}(S))$ is D-dense in $\mathrm{Teich}(S)$. It suffices, then, to show that there are uniform constants $A_1 \ge 1$ and $A_2 \ge 0$ so that

$$\frac{1}{A_1} d_{\mathbf{P}}(P_0,P_1) - A_2 \le d_{\mathrm{WP}}(Q(P_0),Q(P_1)) \le A_1 d_{\mathbf{P}}(P_0,P_1) + A_2.$$

We first show that the map Q is $2D$-Lipschitz. Given P_0 and P_1 such that

$$d_{\mathbf{P}}(P_0,P_1) = 1,$$

P_0 and P_1 differ by a single elementary move. Let $\alpha \in P_0$ and $\beta \in P_1$ be the curves involved in this elementary move, i.e. $P_0 - \alpha = P_1 - \beta$ and $i(\alpha,\beta) = 1$ or 2 depending on whether the component $S_\alpha \subset S - (P_0 - \alpha)$ containing α is a punctured torus or four-times punctured sphere.

Let $Z \in \mathrm{Teich}(S_\alpha)$ be the "square" punctured torus: i.e. Z is obtained by identifying opposite sides of an ideal square in $\mathbb{H}^2$ with order-4 rotational symmetry about the origin in the disk model of $\mathbb{H}^2$. Marking Z so that the common per-

`pics/square.ps` not found

Figure 4. The cover $\widetilde{Z}$ with lifts of α and β passing through the origin.

pendiculars to the opposite sides descend to closed geodesics α and β on Z (see Figure 4), we have by symmetry that the length $\ell_Z(\alpha)$ equals the length $\ell_Z(\beta)$, which is the shortest length of any non-peripheral simple closed curve on Z. In particular, we have

$$\ell_Z(\alpha) < L(S_\alpha)$$

where $L(S_\alpha)$ is the Bers constant for S_α.

Let $(f, W) \in \overline{\mathrm{Teich}(S)}$ be the noded Riemann surface with nodes at each $\gamma \in P - \alpha$, and one non-rigid piece $f|_{S_\alpha} \to Z$. For any $Z' \in \mathrm{Teich}(S)$ sufficiently close to Z, if $Z' \in V(P')$ then P' contains an essential non-peripheral curve in S_α. Taking Z' arbitrarily close to Z, we may conclude that $L(S_\alpha) \leq L(S) = L$. Then for any Riemann surface $Z' \in \mathrm{Teich}(S)$ sufficiently close to $W \in \overline{\mathrm{Teich}(S)}$ we have

$$\ell_{Z'}(\gamma) < L$$

for each $\gamma \in P_0 \cup P_1$. In other words, Z' lies in the intersection $V(P_0) \cap V(P_1)$. Letting Z be the double of the symmetric ideal square described above, a similar argument handles the genus-0 case.

Therefore we may conclude that

$$d_{\mathrm{WP}}(Q(P_0), Q(P_1)) < 2D \quad \text{when} \quad d_{\mathbf{P}}(P_0, P_1) = 1,$$

so by the triangle inequality, Q is $2D$-Lipschitz. To show that for some A_1 and A_2 the inequality

$$\frac{1}{A_1} d_{\mathbf{P}}(P_0, P_1) - A_2 \leq d_{\mathrm{WP}}(Q(P_0), Q(P_1))$$

holds is somewhat more delicate. We break this into a series of lemmas.

Lemma 3.3. *Given $L' > L$, there is an integer $B > 0$ so that given P and P' in $\mathbf{P}(S)$ for which $V_{L'}(P) \cap V_{L'}(P') \neq \varnothing$, we have $d_{\mathbf{P}}(P, P') \leq B$.*

Proof: The hypotheses imply that there is some $X \in \mathrm{Teich}(S)$ so that $\ell_X(\alpha) < L'$ for each $\alpha \in P \cup P'$. By an application of the collar lemma [Bus, Thm. 4.4.6] there is a constant C depending only L' and S so that the total geometric intersection number $i(P, P')$ satisfies

$$i(P, P') \leq C.$$

Let $\mathbf{Tw}(P) \cong \mathbb{Z}^{|P|}$ denote the subgroup of $\mathrm{Mod}(S)$ generated by Dehn twists about the curves in P. Then the function $i(P, .)\colon \mathbf{P}^0(S) \to \mathbb{Z}$ descends to a function

$$i(P, .)\colon \mathbf{P}^0(S)/\mathbf{Tw}(P) \to \mathbb{Z}$$

whose sub level sets are bounded: in other words there are only finitely many equivalence classes

$$\{[P_1], \ldots, [P_c]\} \subset \mathbf{P}^0(S)/\mathbf{Tw}(P)$$

for which $i(P, [P_j]) \leq C$.

Since $d_{\mathbf{P}}(P, .)$ also descends to a function

$$d_{\mathbf{P}}(P, .)\colon \mathbf{P}^0(S)/\mathbf{Tw}(P) \to \mathbb{Z}$$

on $\mathbf{P}^0(S)/\mathbf{Tw}(P)$, we have

$$d_{\mathbf{P}}(P, P') \leq B$$

where

$$B = \max_{j=1,\ldots,c} \{d_{\mathbf{P}}(P, [P_j])\}.$$

∎

Lemma 3.4. *Given $L' > L$, there is an integer $J > 0$, so that if X_t, $t \in [0,1]$, is a unit-length Weil-Petersson geodesic joining X_0 and X_1, then there exist pants decompositions $P_1, \ldots, P_J$ so that $\{X_t\}_{t=0}^1$ lies in the union*

$$V_{L'}(P_1) \cup \ldots \cup V_{L'}(P_J).$$

Proof: Recall from Theorem 2.1 that the sets $V(P) \subset V_{L'}(P)$ cover $\mathrm{Teich}(S)$. Let $P_1, \ldots, P_m$ determine sets $V(P_1), \ldots, V(P_m)$ so that for each $t \in [0,1]$ we have

$$X_t \in V(P_1) \cup \ldots \cup V(P_m).$$

Let

$$d_{L',P} \colon \partial V(P) \to \mathbb{R}_+$$

be the function

$$d_{L',P}(X) = \inf_{Y \in \partial V_{L'}(P)} d_{\mathrm{WP}}(X,Y).$$

We claim that there is an ϵ_0 depending only on L, and L' so that

$$d_{L',P}(X) > \epsilon_0.$$

The function $d_{L',P}(X)$ naturally extends to the metric completion $\overline{\partial V(P)}$ of $\partial V(P)$, and $d_{L',P}(X)$ is invariant under the action of $\mathbf{Tw}(P)$.

Let $\{(\ell_i, \theta_i) \in \mathbb{R}_+^{|P|} \times \mathbb{R}^{|P|}\}$ denote Fenchel-Nielsen coordinates for $\mathrm{Teich}(S)$ adapted to the pants decomposition P. To extend these Fenchel Nielsen coordinates to the completion, we denote by

$$\mathbb{R}_{\geq 0} \times \mathbb{R}/\sim$$

the quotient of $\mathbb{R}_{\geq 0} \times \mathbb{R}$ by the equivalence relation $(0,\theta) \sim (0,\theta')$. Then the completion $\overline{V_{L'}(P)}$ of $V_{L'}(P)$ in $\overline{\mathrm{Teich}(S)}$ admits *extended* Fenchel-Nielsen coordinates

$$\overline{V_{L'}(P)} = \{(\ell_i, \theta_i) \in \mathbb{R}^+ \times \mathbb{R}/\sim \ \mid \ell_i \leq L',\ i = 1, \ldots, |P|\}$$

where each point with $\ell_j = 0$ for some j lies in the completion. The extended isometric action of $\mathbf{Tw}(P)$ on $\overline{\mathrm{Teich}(S)}$ is cocompact on $\overline{V_{L'}(P)}$, since $\mathbf{Tw}(P)$ preserves each length coordinate and acts by translations on each twist coordinate.

In these extended Fenchel-Nielsen coordinates, the completion $\overline{\partial V_{L'}(P)}$ of the boundary $\partial V_{L'}(P)$ is the locus of coordinates for which $\ell_j = L'$ for some $j \in 1, \ldots, |P|$. Thus

$$\overline{\partial V_{L'}(S)}/\mathbf{Tw}(P)$$

is a closed subset of the compact set $\overline{V_{L'}(P)}/\mathbf{Tw}(P)$ and is thus compact.

Since the quotients

$$\overline{\partial V_{L'}(P)}/\mathbf{Tw}(P) \quad \text{and} \quad \overline{\partial V(P)}/\mathbf{Tw}(P)$$

are disjoint compact subsets of $\overline{V_{L'}(P)}/\mathbf{Tw}(P)$, it follows that the function

$$\mathcal{I}_{L'} \colon \mathbf{P}^0(S) \to \mathbb{R}_+$$

given by

$$\mathcal{I}_{L'}(P) = \inf_{X \in \partial V(P)} d_{L',P}(X)$$

is positive. But $\mathcal{I}_{L'}$ is $\mathrm{Mod}(S)$-invariant, so it descends to a function

$$\mathcal{I}_{L'} \colon \mathbf{P}^0(S)/\mathrm{Mod}(S) \to \mathbb{R}_+.$$

Since $\mathbf{P}^0(S)/\mathrm{Mod}(S)$ is finite, we may set ϵ_0 equal to the infimum of $\mathcal{I}_{L'}([P])$ over the finite number of equivalence classes $[P] \in \mathbf{P}^0(S)/\mathrm{Mod}(S)$.

If X_{t_0} lies in $V(P)$, then, X_t lies in $V_{L'}(P)$, provided t lies in $(t_0 - \epsilon_0, t_0 + \epsilon_0)$. It follows that after setting J equal to the least integer greater than $2/\epsilon_0$, we may select from the pants decompositions $P_1, \ldots, P_m$ pants decompositions $P_1, \ldots, P_J$ (possibly with repetition) so that

$$X_t \in V_{L'}(P_1) \cup \ldots \cup V_{L'}(P_J)$$

for each $t \in [0, 1]$.

∎

To complete the proof of Theorem 3.2, let X_t be the Weil-Petersson geodesic joining arbitrary distinct Riemann surfaces X and Y in $\mathrm{Teich}(S)$. Let P_X and P_Y be pants decompositions for which X lies in $V(P_X)$ and Y lies in $V(P_Y)$. Let $I(P) \subset [0, 1]$ denote the values of t for which $X_t \in V_{2L}(P)$. By convexity of $V(P)$ (Proposition 2.2) each $I(P)$ is an interval.

Taking $L' = 2L$, Lemma 3.4 provides a $J > 0$ and a sequence $\{P_j\}_{j=0}^N \in \mathbf{P}(S)$ so that

- $X(t)$ is covered by the union $\cup_j I(P_j)$,
- the least upper bound of $I(P_j)$ lies in $I(P_{j+1})$, and
- $N \leq J(d_{\mathrm{WP}}(X, Y) + 1)$.

Thus, we have

$$\frac{N}{J} - 1 \leq d_{\mathrm{WP}}(X, Y). \tag{3.2}$$

Moreover, for successive pants decompositions P_j, P_{j+1}, we have

$$V_{2L}(P_j) \cap V_{2L}(P_{j+1}) \neq \varnothing,$$

so applying Lemma 3.3 with $L' = 2L$, we have a $B > 0$ for which

$$d_{\mathbf{P}}(P_X, P_Y) \leq BN. \tag{3.3}$$

Combining equations 3.2 and 3.3, we have

$$\frac{d_{\mathbf{P}}(P_X, P_Y)}{BJ} - 1 \leq d_{\mathrm{WP}}(Q(P_X), Q(P_Y)), \tag{3.4}$$

where B and J depend only on L which depends only on S. Setting $A_1 = BJ$ and $A_2 = 1$ concludes the proof of Theorem 3.2.

∎

4. Bounding the core volume from below

To simplify notation, let

$$C(X, Y) = \mathrm{core}(Q(X, Y))$$

and recall that $\mathrm{vol}(X, Y)$ denotes the convex core volume $\mathrm{vol}(\mathrm{core}(Q(X, Y)))$. In this section we prove

Theorem 4.1. *Given S, there are constants $K_1 > 1$ and $K_2 > 0$ so that*

$$\frac{1}{K_1} d_{\mathrm{WP}}(X, Y) - K_2 \leq \mathrm{vol}(X, Y).$$

The proof is given as a series of lemmas.

Fix attention on a given quasi-Fuchsian manifold $Q(X,Y)$. Given a constant $L_0 > 0$, let $\mathcal{S}_{<L_0} \subset \mathcal{S}$ denote the set of isotopy classes

$$\mathcal{S}_{<L_0} = \{\alpha \in \mathcal{S} \mid \ell_{Q(X,Y)}(\alpha) < L_0\}.$$

Theorem 4.1 will follow from a linear lower bound on $\mathrm{vol}(X,Y)$ given in terms of the size of $\mathcal{S}_{<L}$ (Lemma 4.8) and the following lemma.

Lemma 4.2. *Let P_X and P_Y be pants decompositions so that $X \in V(P_X)$ and $Y \in V(P_Y)$. Then there is a constant K depending only on S so that*

$$d_{\mathbf{P}}(P_X, P_Y) \leq K|\mathcal{S}_{<L}|.$$

The lemma will follow from the following general result on paths in $\mathbf{P}(S)$ that are built out of a given collection of curves in $\mathcal{S}$. For reference, let

$$\pi_{\mathcal{S}} \colon \mathbf{P}^0(S) \to \mathcal{S}$$

denote the projection that assigns to each P the collection of curves used to build it.

Lemma 4.3. *Let $k \in \mathbb{N}$ and let $g = \{P_I = P_0, \ldots, P_N = P_T\} \subset \mathbf{P}(S)$ be a sequence of pants decompositions with the property that $d_{\mathbf{P}}(P_j, P_{j+1}) < k$. Let $\mathcal{S}_g \subset \mathcal{S}$ denote the image of g under the projection $\pi_{\mathcal{S}}$. There is a constant $K_0 > 0$ depending on k and S so that*

$$d_{\mathbf{P}}(P_I, P_T) < K_0|\mathcal{S}_g|.$$

The complex of curves. The graph $\mathbf{P}(S)$ is related to the *complex of curves* $\mathcal{C}(S)$, introduced by W. Harvey [Har]. To prove Lemma 4.3 we describe recent work of Masur and Minksy on $\mathcal{C}(S)$. The main result of [MM1] shows that $\mathcal{C}(S)$ is in fact a *Gromov hyperbolic* metric space with the metric obtained by making each simplex a standard Euclidean simplex. Its sequel [MM2] introduces a theory of *hierarchies* of so-called "tight geodesics" in $\mathcal{C}(S)$ and in sub-complexes $\mathcal{C}(Y)$ for essential subsurfaces $Y \subset S$. Such hierarchies and their hyperbolicity properties play an integral role in our control of volume.

To describe the topological type of S, we let

$$d(S) = \dim_{\mathbb{C}}(\mathrm{Teich}(S)) = 3g - 3 + n$$

where S has genus g with n boundary components. We consider only those surfaces S for which $\mathrm{int}(S)$ admits a hyperbolic structure (so $d(S) > 0$).

The *complex of curves* $\mathcal{C}(S)$ is a simplicial complex with 0-skeleton $\mathcal{S}$, and higher dimensional simplices described as follows:

- for $d(S) > 1$ and $k \geq 1$, k-simplices of $\mathcal{C}(S)$ span $k+1$-tuples $\alpha_1, \ldots, \alpha_{k+1}$ of vertices for which $i(\alpha_i, \alpha_j) = 0$, and
- if $d(S) = 1$, $\mathcal{C}(S)$ is a 1-complex whose edges join vertices α and α' in $\mathcal{C}(S)$ that intersect minimally; i.e. $\mathcal{C}(S) = \mathbf{P}(S)$.

Given an essential subsurface $Y \subset S$ with $d(Y) \geq 2$, the curve complex $\mathcal{C}(Y)$ is naturally a subcomplex of $\mathcal{C}(S)$. Given a set W let $\mathcal{P}(W)$ denote its power set, i.e. the set of all subsets of W. Masur and Minsky define a projection

$$\pi_Y \colon \mathcal{C}(S) \to \mathcal{P}(\mathcal{C}(Y)),$$

by setting $\pi(\alpha) = \alpha$ if $\alpha \in \mathcal{C}(Y)$, and taking

$$\pi_Y(\alpha) = \bigcup_{\alpha' \subset \alpha \cap Y} \partial \mathcal{N}(\alpha' \cup \partial_{\alpha'} Y)$$

where α' is an arc of essential intersection of α with Y, $\partial_{\alpha'} Y \subset \partial Y$ is the components of ∂Y that α' meets in its endpoints, and $\mathcal{N}(.)$ denotes a regular neighborhood of their union (see [MM2, Sec. 2]).

When A and B are two subsets of $\mathcal{C}(Y)$, [MM2] defines a coarse distance $d_Y(A, B)$ by taking the diameter

$$d_Y(A, B) = \operatorname{diam}_{\mathcal{C}(Y)}(A \cup B),$$

in $\mathcal{C}(Y)$ of A and B. Note that while $d_Y(.,.)$ is more a diameter than a distance when A and B are close, it gives a useful notion of distance between sets of bounded diameter and does satisfy the triangle inequality.

By [MM2, Lem. 2.3] the projection π_Y has a Lipschitz property: if Δ is a simplex in $\mathcal{C}(S)$ so that Δ intersects Y, then we have $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\Delta)) \le 2$. If P_I and P_T are two subsets of $\mathcal{C}(S)$, letting $\pi_Y(P_I) = \cup_{\alpha \in P_I} \pi_Y(\alpha)$, and likewise for P_T, then the *projection distance* $d_Y(P_I, P_T)$ between P_I and P_T (or *distance in* Y) is defined by

$$d_Y(P_I, P_T) = d_Y(\pi_Y(P_I), \pi_Y(P_T)).$$

In particular, if P and P' are pants decompositions that differ by a single elementary move, then we have $d_Y(P, P') \le 4$ (see [MM2, Lem. 2.5]).

A central theorem we will use is the following:

Theorem 4.4 (Thm. 6.12 of [MM2])**.** *There is a constant $M_0(S)$ so that given $M > M_0$ there exist c_0 and c_1 so that if P_I and P_T are pants decompositions in $\mathbf{P}(S)$ then we have*

$$\frac{1}{c_0} d_{\mathbf{P}}(P_I, P_T) - c_1 \le \sum_{\substack{Y \subseteq S \\ d_Y(P_I, P_T) > M}} d_Y(P_I, P_T) \le c_0 d_{\mathbf{P}}(P_I, P_T) + c_1$$

where the sum is taken over all non-annular essential subsurfaces $Y \subseteq S$ satisfying $d_Y(P_I, P_T) > M$.

We apply this result to prove Lemma 4.3. Our argument is quite similar to that of [MM2, Thm. 6.10], where it is shown that a given pants decomposition along an elementary move sequence can contribute to progress in only boundedly many projections to subsurfaces simultaneously. We seek the analogous statement for a single curve occuring in pants decompositions joining P_I to P_T.

Proof: *(of Lemma 4.3).* To prove the lemma, we will relate the sum of the projections to the size of $\mathcal{S}_g$. To do this, we note that when the projection distance $d_Y(P_I, P_T)$ is large, there must be a definite portion of the projection of g to Y that is far from both $\pi_Y(P_I)$ and $\pi_Y(P_T)$ in $d_Y(.,.)$; this follows from the triangle inequality for d_Y and the fact that elementary moves in $\mathbf{P}(S)$ make Lipschitz progress as measured by d_Y.

We argue that a given curve α can contribute only to a bounded amount of progress in boundedly many different subsurfaces. Precisely, let Y and Z be two essential, intersecting, non-annular subsurfaces of S, neither of which is contained in the other. A lemma of Masur and Minsky [MM2, Lem. 6.11] enforces a partial ordering "$\prec$" on such subsurfaces with respect to the pants decompositions P and

P', provided the projection distances $d_Y(P,P')$ and $d_Z(P,P')$ are greater than a constant M_2 depending only on S. Taking M_2 to be the constant of [MM2, Lem. 6.2] with the same name, we say the subsurfaces Y and Z are (P,P')-*ordered* if we have

$$d_Y(P,P') > M_2 \quad \text{and} \quad d_Z(P,P') > M_2.$$

We rephrase [MM2, Lem. 6.11] as follows.

Lemma 4.5. *There is a constant M_3 depending only on S so that if Y and Z are (P,P')-ordered then one of two cases obtains. Either $Y \prec Z$, and we have*

$$d_Y(\partial Z, P') < M_3 \quad \textit{and} \quad d_Z(P, \partial Y) < M_3,$$

or $Z \prec Y$, and we have

$$d_Z(\partial Y, P') < M_3 \quad \textit{and} \quad d_Y(P, \partial Z) < M_3.$$

Proof: If Y and Z are (P,P')-ordered in the above sense, then by [MM2, Lem. 6.2] they appear as *domains* $Y = D(h)$ and $Z = D(k)$ supporting *tight geodesics* $h \subset \mathcal{C}(Y)$ and $k \subset \mathcal{C}(Z)$ in any *hierarchy* H (without annuli [MM2, Sec. 8]) joining $P = I(H)$ to $P' = T(H)$. The condition that Y and Z intersect and are non-nested guarantees that h and k are *time-ordered* [MM2, Lem. 4.18] (in the sense of [MM2, Defn. 4.16]). The lemma then follows from an application of [MM2, Lem. 6.11] where $Y \prec Z$ represents the case $h \prec_t k$ and $Z \prec Y$ represents the case $k \prec_t h$.

∎

Let $M_4 = M_2 + 2M_3 + 4$, and let $M = \max\{4M_4, M_0\}$. Consider an essential subsurface $Y \subseteq S$ for which $d_Y(P_I, P_T) > M$. As in the proof of [MM2, Lem. 6.10] let J_Y denote the subset of $[1,N]$ for which if $i \in J_Y$ then P_i is "deep" in the projection to Y: i.e.

$$d_Y(P_I, P_i) > M_4 \ \text{ and } \ d_Y(P_i, P_T) > M_4.$$

Given a subset $A \subset [1,N]$, we denote by

$$\|A\|_Y = \operatorname{diam}_{\mathcal{C}(Y)}(\{\pi_Y(P_i) \mid i \in A\})$$

the diameter of the projection of the pants decompositions with indices in A to the curve complex $\mathcal{C}(Y)$.

Given $\alpha \in \mathcal{S}_g$ for which $\pi_Y(\alpha) \neq \varnothing$, we denote by $J_Y(\alpha) \subset J_Y$ the subset for which if $i \in J_Y(\alpha)$ then α lies in P_i. We make three observations for later reference:

I. By the Lipschitz property for π_Y, we have $\|J_Y(\alpha)\|_Y \leq 4$.
II. If i lies in J_Y then there is some $\alpha \in \mathcal{S}_g$ so that $i \in J_Y(\alpha)$.
III. Since $d_{\mathbf{P}}(P_j, P_{j+1}) < k$, we have $d_Y(P_j, P_{j+1}) < 4k$.

Let $Y \subseteq S$ and $Z \subseteq S$ be two non-annular intersecting subsurfaces neither of which is contained in the other so that each contributes to the sum of Theorem 4.4: i.e. we have

$$d_Y(P_I, P_T) > M \quad \text{and} \quad d_Z(P_I, P_T) > M.$$

This assumption guarantees, in particular, that Y and Z are (P_I, P_T)-ordered. We make the following claim:

($*$) *If $J_Y(\alpha)$ is non-empty, then $J_Z(\alpha)$ must be empty.*

Arguing by contradiction, assume

$$J_Y(\alpha) \neq \varnothing \neq J_Z(\alpha).$$

We have

$$\pi_Y(\alpha) \neq \varnothing \neq \pi_Z(\alpha)$$

so if $i \in J_Y(\alpha)$ and $j \in J_Z(\alpha)$ then $\operatorname{diam}_Y(\pi_Y(P_i)) \leq 2$, $\operatorname{diam}_Y(\pi_Y(P_j)) \leq 2$ so it follows that $d_Y(P_i, P_j) \leq 4$, since α lies in P_i and in P_j. The same conclusion holds with Z in place of Y.

Since i lies in $J_Y(\alpha)$ we have $d_Y(P_I, P_i) > M_4$ and $d_Y(P_i, P_T) > M_4$, so it follows that

$$d_Y(P_I, P_j) \geq M_4 - 4 \quad \text{and} \quad d_Y(P_j, P_T) \geq M_4 - 4.$$

As j lies in $J_Z(\alpha)$ we have

$$d_Z(P_I, P_j) \geq M_4,$$

and since $M_4 - 4 > M_2$, it follows that Y and Z are also (P_I, P_j)-ordered. Let $\prec_j$ denote the (P_I, P_j)-ordering and assume without loss of generality that $Y \prec_j Z$. Then applying Lemma 4.5 we have

$$d_Y(\partial Z, P_j) < M_3.$$

Since Y and Z are also (P_I, P_T)-ordered, we may first assume that $Y \prec Z$. Then Lemma 4.5 gives $d_Y(\partial Z, P_T) < M_3$ which implies that

$$d_Y(P_j, P_T) < 2M_3,$$

contradicting the assumption that $j \in J_Z(\alpha)$. If on the other hand we have $Z \prec Y$, then Lemma 4.5 gives $d_Y(P_I, \partial Z) < M_3$ from which we conclude

$$d_Y(P_I, P_j) < 2M_3,$$

which contradicts the same assumption. Thus, either $J_Y(\alpha)$ or $J_Z(\alpha)$ must be empty, and the claim $(*)$ is proven.

Applying observations **(I)** and **(II)** above, if $\mathcal{S}_Y = \{\alpha \in \mathcal{S}_g \mid J_Y(\alpha) \neq \varnothing\}$, then we have

$$\|J_Y\|_Y \leq 4|\mathcal{S}_Y|.$$

There is a uniform bound s depending only on S to the size of any collection of subsurfaces any pair of which is disjoint or nested (see [MM2, Lem. 6.10, proof]) so by our claim $(*)$, the number of Y for which $J_Y(\alpha)$ can be non-empty is bounded by s. Thus we have

$$\sum_{Y \subseteq S, J_Y \neq \varnothing} \|J_Y\|_Y \leq 4s|\mathcal{S}_g|.$$

Applying observation **(III)**, we have $d_Y(P_j, P_{j+1}) < 4k$. Thus, for each Y satisfying

$$d_Y(P_I, P_T) > M$$

the set J_Y is in particular non-empty, and we have

$$d_Y(P_I, P_T) - 2M_4 \leq 4k\|J_Y\|_Y.$$

But since $4M_4 < M$ we have $2M_4 \leq 4k\|J_Y\|_Y$ and thus

$$d_Y(P_I, P_T) \leq 8k\|J_Y\|_Y.$$

Since $M = \max\{M_0, 4M_4\}$, applying Theorem 4.4 there are constants c_0 and c_1 so that we have

$$\frac{1}{c_0} d_{\mathbf{P}}(P_I, P_T) - c_1 \leq \sum_{\substack{Y \subseteq S \\ d_Y(P_I, P_T) > M}} d_Y(P_I, P_T) \leq 32sk|\mathcal{S}_g|.$$

Since $|\mathcal{S}_g|$ is always at least $d(S)$, we may combine all of the above constants into a single K_0 for which

$$d_{\mathbf{P}}(P_I, P_T) \leq K_0 |\mathcal{S}_g|.$$

∎

To prove Lemma 4.2, we will apply Lemma 4.3 to a sequence of pants decompositions $\{P_X = P_0, \ldots, P_N = P_Y\}$ so that $\pi_{\mathcal{S}}(P_j) \subset \mathcal{S}_{<L}$ for each j, and so that $d_{\mathbf{P}}(P_j, P_{j+1})$ is bounded by an a priori constant. The existence of such a sequence is provided by an interpolation of 1-Lipschitz homotopy equivalences of hyperbolic surfaces into $Q(X, Y)$ that pass from one side of the convex core to the other. The existence of such an interpolation follows from work of R. Canary on *simplicial hyperbolic surfaces* which we now describe.

Simplicial hyperbolic surfaces. Let $\mathrm{Sing}_k(S)$ denote the finite-area marked *singular* hyperbolic structures on S: complete finite area hyperbolic surfaces Z with at most k cone singularities, each with cone-angle at least 2π, equipped with marking homeomorphisms $h\colon \mathrm{int}(S) \to Z$ up to marking-preserving isometry. Roughly speaking, a *simplicial hyperbolic surface* is a path-isometric mapping from a singular hyperbolic surface to a hyperbolic 3-manifold that is totally geodesic in the complement of a "triangulation." We now make this notion precise.

Let $V = \{v_1, \ldots, v_p\}$ be a finite subset of S. Following Hatcher [Hat], an *essential arc* α in (S, V) is an embedded arc meeting $\partial S \cup V$ only in its endpoints, which lie in V. A collection $\{\alpha_0, \ldots, \alpha_k\}$ of essential arcs in (S, V) that are pairwise embedded and non-isotopic *rel*-endpoints is called a *curve system*. Let $\mathcal{A}(S, V)$ denote the simplicial complex whose k-simplices $[\alpha_0, \ldots, \alpha_k]$ are curve systems with curves $\{\alpha_0, \ldots, \alpha_k\}$ with faces given by k-tuples of curves in $\{\alpha_0, \ldots, \alpha_k\}$.

If V contains a point in each boundary component of the compact surface S, then a *triangulation* of S is a maximal curve system in $\mathcal{A}(S, V)$. Likewise, we may view the interior $\mathrm{int}(S)$ of S as a "punctured surface" by collapsing each boundary component $\gamma \subset \partial S$ to a point v_γ to obtain a surface R. Then we have

$$\mathrm{int}(S) = R - \{v_\gamma \mid \gamma \subset \partial S\}.$$

If V is a subset of R containing $\cup_\gamma v_\gamma$, then a *triangulation* of $\mathrm{int}(S)$ is the restriction of a maximal curve system in $\mathcal{A}(R, V)$ restricted to $R - \{v_\gamma \mid \gamma \subset \partial S\} = \mathrm{int}(S)$. Note that in each definition, an edge may have its boundary vertices identified and a face may have boundary edges identified.

The main result of [Hat] guarantees that any two triangulations in $\mathcal{A}(S, V)$ are related by a finite sequence of elementary moves (see Figure 5).

Let T be a triangulation of $\mathrm{int}(S)$ in the above sense, and let $h\colon \mathrm{int}(S) \to Z$ be a singular hyperbolic surface for which h is isotopic to a map with the property that each cone singularity of Z lies in the image of a vertex of T. Isotope h to send each edge of T to its geodesic representative *rel*-endpoints on Z (if an edge e terminates at a puncture, h should send e to a geodesic arc asymptotic to the corresponding cusp of Z). Then if N is a hyperbolic 3-manifold and there is a

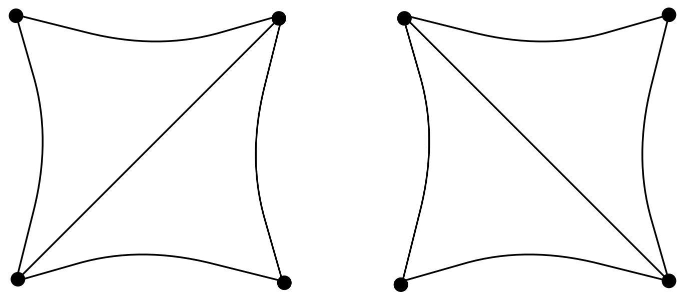

Figure 5. Elementary moves on triangulations.

path-isometry $g\colon Z \to N$ that is a local isometry on $Z - T$, then we call the pair (g, Z) a *simplicial hyperbolic surface in* N with *associated triangulation* T.

Often the construction goes in the other direction: given a triangulation T of $\mathrm{int}(S)$ with one vertex $v \in \mathrm{int}(S)$ and at least one edge e so that $e \cup v$ forms a closed loop γ, we can straighten any smooth, proper, incompressible map g' of $\mathrm{int}(S)$ to N to a simplicial hyperbolic surface with associated triangulation T. First we straighten g' so it maps γ to its geodesic representative. Then, straightening g' on the edges of T *rel*-endpoints (possibly ideal endpoints) and then on faces of T, we obtain a map $g\colon \mathrm{int}(S) \to N$. The pull-back metric from N determines a singular hyperbolic surface Z with a cone singularity at v; since v lies on a closed geodesic that is mapped to a closed geodesic in N, the cone angle at v is at least 2π. The result is a simplicial hyperbolic surface (g, Z) in N with associated triangulation T. In this case we say that (g, Z) is *adapted* to γ.

Let $(f\colon S \to N) \in AH(S)$. Then we denote by $\mathcal{SH}_k(N)$ the *marking preserving simplicial hyperbolic surfaces in* N *with at most* k *cone-singularities*, namely, simplicial hyperbolic surfaces (g, Z), with $(h\colon \mathrm{int}(S) \to Z) \in \mathrm{Sing}_k(S)$, so that $g \circ h$ is homotopic to f. If σ is a simplex in $\mathcal{C}(S)$ with vertices $\alpha_1, \ldots, \alpha_p$, then we say a simplicial hyperbolic surface $(g, Z) \in \mathcal{SH}_k(N)$ *realizes* σ if g maps each α_i isometrically to its geodesic representative in N.

Recall that a manifold $(f\colon S \to N) \in AH(S)$ has an *accidental parabolic* if there is a non-peripheral element $\gamma \in \pi_1(S)$ so that $f_*(\gamma)$ is a parabolic element of $\mathrm{Isom}^+(\mathbb{H}^3)$. Applying Hatcher's theorem [Hat] allowing one to connect triangulations by elementary moves, Canary proves the following (see [Can3, §5, §6]; compare [Br1, §3] and [Min, §4]):

Theorem 4.6 (Canary)**.** *Let $N \in AH(S)$ have no accidental parabolics, and let (g_1, Z_1) and (g_2, Z_2) lie in $\mathcal{SH}_1(N)$ where (g_1, Z_1) is adapted to α and (g_2, Z_2) is adapted to β. Then there is a continuous family $(g_t\colon Z_t \to N) \subset \mathcal{SH}_2(N)$, $t \in [1, 2]$.*

Using such an interpolation, we now give the proof of Lemma 4.2.

Proof: *(of Lemma 4.2).* Given the quasi-Fuchsian manifold $Q(X, Y)$, let P_X and P_Y denote pants decompositions for which X lies in $V(P_X)$ and Y lies in $V(P_Y)$. Then by a theorem of Bers ([Brs2, Thm. 3], [Mc1, Prop. 6.4]), we have

$$\ell_{Q(X,Y)}(\alpha) < 2L$$

for each curve α in $P_X \cup P_Y$.

Let Z_X and Z_Y denote simplicial hyperbolic surfaces realizing P_X and P_Y in $Q(X,Y)$ and let T_X and T_Y denote their associated triangulations. Let $Z_X^{\mathrm{h}} \in \mathrm{Teich}(S)$ be the hyperbolic surface conformally equivalent to Z_X, and let Z_Y^{h} be the hyperbolic surface conformally equivalent to Z_Y. Finally, let P_I and P_T be pants decompositions so that $Z_X^{\mathrm{h}} \in V(P_I)$ and $Z_Y^{\mathrm{h}} \in V(P_T)$.

The next step will be to interpolate simplicial hyperbolic surfaces between Z_X and Z_Y and estimate the minimum number of sets $V(P_j)$ the corresponding conformally equivalent hyperbolic representatives in $\mathrm{Teich}(S)$ intersect. To show we have not sacrificed too much distance in the pants graph, we prove the following:

Lemma 4.7. *There is a constant $B' > 0$ depending only on S so that*

$$\max\{d_{\mathbf{P}}(P_X, P_I), d_{\mathbf{P}}(P_Y, P_T)\} < B'.$$

Proof: We have

$$\ell_{Z_X}(\gamma) < 2L$$

for each $\gamma \in P_X$. By a lemma of Ahlfors [Ah] we have

$$\ell_{Z_X^{\mathrm{h}}}(\beta) \geq \ell_{Z_X}(\beta)$$

for each $\beta \in \mathcal{S}$. Thus, we have

$$\ell_{Z_X}(\alpha) < 2L$$

for each $\alpha \in P_X \cup P_I$. Since curves in $P_X \cup P_I$ are realized with bounded length on the simplicial hyperbolic surface $Z_X \in \mathcal{SH}_k(Q(X,Y))$, we may apply [Br1, Lem. 3.3] to find a $C' > 0$ depending only on L so that

$$i(P_X, P_I) < C'.$$

Arguing similarly for Z_Y, we have

$$i(P_Y, P_T) < C'.$$

Applying the proof of Theorem 3.3, we have a $B' > 0$ depending only on C' for which

$$\max\{d_{\mathbf{P}}(P_X, P_I), d_{\mathbf{P}}(P_Y, P_T)\} < B'.$$

■

To complete the proof of Lemma 4.2, we seek a continuous family

$$(h_t \colon Z_t \to Q(X,Y)) \subset \mathcal{SH}_k(Q(X,Y))$$

of simplicial hyperbolic surfaces in $Q(X,Y)$ interpolating between (h_X, Z_X) and (h_Y, Z_Y). We first connect (h_X, Z_X) and (h_Y, Z_Y) to simplicial hyperbolic surfaces (h'_X, Z'_X) and (h'_Y, Z'_Y) adapted to single curves $\alpha \in P_X$ and $\beta \in P_Y$ by continuous families, and then apply the interpolation arguments of Canary.

This is easily done by collapsing edges of T_X that join distinct vertices of T_X down to a single vertex: if e is such an edge adjacent to a vertex v on α, then we may effect such a collapsing by 'dragging h_X along $h_X(e)$' (see [Can3, §5]). Precisely, if v' is the other vertex in ∂e, we construct a homotopy of (h_X, Z_X) to a new simplicial hyperbolic surface by pulling the image $h_X(v')$ along the geodesic segment $h_X(e)$ to $h_X(v)$ and pulling the edges and faces adjacent to v along with it while keeping the triangulation straight: the image of each triangle is required to lift to the convex hull of its vertices throughout the homotopy.

It is easy to check that under such a collapsing the cone angles at the vertices remain at least 2π, and the number of vertices in T is reduced by 1. We successively collapse edges joining v to different vertices until we are left with a triangulation with the single vertex v, and a simplicial hyperbolic surface $(h'_X, Z'_X) \in \mathcal{SH}_1(Q(X,Y))$ adapted to α.

We perform analogous collapsings on the associated triangulation for (h_Y, Z_Y) to obtain the simplicial hyperbolic surface $(h'_Y, Z'_Y) \in \mathcal{SH}_1(Q(X,Y))$ adapted to β. By Theorem 4.6, we may interpolate between (h'_X, Z'_X) and (h'_Y, Z'_Y) by a continuous family of simplicial hyperbolic surfaces, so we have the desired continuous family

$$(h_t \colon Z_t \to C(X,Y)) \subset \mathcal{SH}_k(Q(X,Y)), \quad t \in [0,1]$$

so that $(h_0, Z_0) = (h_X, Z_X)$ and $(h_1, Z_1) = (h_Y, Z_Y)$.

The singular hyperbolic structures Z_t determine a continuous path

$$(h^{\mathrm{h}}_t \colon \mathrm{int}(S) \to Z^{\mathrm{h}}_t) \subset \mathrm{Teich}(S),$$

where as before Z^{h}_t is the finite-area hyperbolic structure on S in the same conformal class as Z_t. Since Z_t and Z^{h}_t represent metrics on the same underlying surface $\mathrm{int}(S)$, we have a natural continuous family of 1-Lipschitz mappings

$$\hat{h}_t \colon Z^{\mathrm{h}}_t \to C(X,Y)$$

of hyperbolic surfaces $Z^{\mathrm{h}}_t \in \mathrm{Teich}(S)$ into $Q(X,Y)$ so that for each t, $\hat{h}_t$ factors through the simplicial hyperbolic surface $h_t \colon Z_t \to C(X,Y)$.

There are pants decompositions $P_1, \ldots, P_N$ (possibly with repetition) that determine an open cover $\{U_j\}_{j=0}^N$ of $[0,1]$ so that if $t \in U_j$ then Z^{h}_t lies in $V(P_j)$, and so that $U_j \cap U_{j+1} \neq \varnothing$ for each $j = 0, \ldots, N-1$. Applying Lemma 3.3, the sequence of pants decompositions

$$g = \{P_I = P_0, \ldots, P_N = P_T\}$$

satisfies the hypotheses of Lemma 4.3 with $k = B$. Applying Lemma 4.3, we have a K_0 so that

$$d_{\mathbf{P}}(P_I, P_T) \leq K_0 |\mathcal{S}_g|.$$

Since the non-empty set $\mathcal{S}_g$ is a subset of $\mathcal{S}_{<L}$, and Lemma 4.7 guarantees

$$d_{\mathbf{P}}(P_I, P_T) \geq d_{\mathbf{P}}(P_X, P_Y) - 2B',$$

we may combine constants to obtain a K for which

$$d_{\mathbf{P}}(P_X, P_Y) \leq K |\mathcal{S}_{<L}|$$

proving the lemma.

∎

Given a hyperbolic 3-manifold M, we let $\mathcal{G}_{<L}(M)$ denote the set of homotopy classes of closed geodesics in M with length bounded above by a constant $L > 0$. We use the contraction $\mathcal{G}_{<L} = \mathcal{G}_{<L}(M)$ when the manifold M is understood. The next lemma shows that the size $|\mathcal{G}_{<L}|$ of $\mathcal{G}_{<L}$ provides a lower bound for the convex core volume of a hyperbolic 3-manifold in a general context.

Lemma 4.8. *Let M be a geometrically finite hyperbolic 3-manifold with ∂M incompressible, and let* $\mathrm{vol}(M)$ *denote its convex core volume. Then there is a constant $C_1 > 1$ depending only on L and $C_2 > 0$ depending only on $\chi(\partial M)$ for which*

$$\frac{|\mathcal{G}_{<L}|}{C_1} - C_2 < \mathrm{vol}(M).$$

Proof: Let $\epsilon > 0$ be less than the minimum of $L/2$ and the 3-dimensional Margulis constant. Let $\mathcal{V}$ be any maximal set of points in the ϵ-thick part $\mathrm{core}(M)_{\geq \epsilon}$ of the convex core $\mathrm{core}(M)$ for which points in $\mathcal{V}$ are separated by a distance at least $\epsilon/2$. Letting $B(x,R)$ denote the ball of radius R about x in M, it follows that $B(x,\epsilon/4)$ is embedded in the ϵ-neighborhood $\mathcal{N}_\epsilon(\mathrm{core}(M)_{\geq\epsilon})$ of the ϵ-thick part of M, and

$$B(x,\epsilon/4) \cap B(x',\epsilon/4) = \varnothing$$

for $x \neq x'$ in $\mathcal{V}$. By maximality, however, we have

$$\mathrm{core}(M)_{\geq\epsilon} \subset \bigcup_{x\in\mathcal{V}} B(x,\epsilon/2).$$

Each isotopy class $\beta \in \mathcal{G}_{<L}$ has a representative $\beta^\star \subset \mathrm{core}(M)_{\geq\epsilon}$ with arclength less than L. Since the $\epsilon/2$-balls about points in $\mathcal{V}$ cover $\mathrm{core}(M)_{\geq\epsilon}$, each $\beta^\star$ intersects $B(x,\epsilon/2)$ for some $x \in \mathcal{V}$. Given $x \in \mathcal{V}$, let $\mathcal{A}_x \subset \mathcal{G}_{<L}$ denote the set

$$\mathcal{A}_x = \{\beta \in \mathcal{G}_{<L} \mid \beta^\star \cap B(x,\epsilon/2) \neq \varnothing\}.$$

Lifting to the universal cover so that x lifts to the origin $0 \in \mathbb{H}^3$, the elements $\beta \in \mathcal{A}_x$ determine pairwise disjoint translates of the ball $B(0,\epsilon/2) \subset \mathbb{H}^3$ lying within the ball $B(0, L+2\epsilon) \in \mathbb{H}^3$. It follows that the number of elements in each $\mathcal{A}_x$ satisfies

$$|\mathcal{A}_x| < \frac{\mathrm{vol}(B(0,L+2\epsilon))}{\mathrm{vol}(B(0,\epsilon/2))}$$

which we set equal to C_0.

Since every $\beta \in \mathcal{G}_{<L}$ lies in some $\mathcal{A}_x$, we have

$$\frac{|\mathcal{G}_{<L}|}{C_0} \leq |\mathcal{V}|.$$

Since the balls of radius $\epsilon/4$ about points $x \in \mathcal{V}$ are embedded and pairwise disjoint in $\mathcal{N}_\epsilon(\mathrm{core}(M)_{\geq\epsilon})$, we have the lower bound

$$\frac{|\mathcal{G}_{<L}|}{C_0} \cdot \mathrm{vol}(B(0,\epsilon/4)) \leq \mathrm{vol}_\epsilon(M)$$

where $\mathrm{vol}_\epsilon(M) = \mathrm{vol}(\mathcal{N}_\epsilon(\mathrm{core}(M)))$. There is a constant $K_\epsilon > 0$ depending only on ϵ and M so that

$$\mathrm{vol}_\epsilon(M) - \mathrm{vol}(M) < K_\epsilon$$

(see e.g. [Can2, Lem. 8.2], [Th1, 8.12.1]). The lemma follows by setting $C_1 = C_0/\mathrm{vol}(B(0,\epsilon/4))$ and $C_2 = K_\epsilon$.

∎

Proof: *(of Theorem 4.1).* Since $\mathcal{S}_{<L}$ is a subset of $\mathcal{G}_{<L}(Q(X,Y))$, we may combine Lemma 4.2 with Lemma 4.8 to obtain

$$\frac{d_{\mathbf{P}}(P_X, P_Y)}{K \cdot C_1} - C_2 \leq \mathrm{vol}(X,Y). \tag{4.5}$$

Applying Theorem 3.2 we have

$$\frac{d_{\mathrm{WP}}(X,Y)}{A_1 \cdot K \cdot C_1} - \frac{A_2}{K \cdot C_1} - C_2 \leq \mathrm{vol}(X,Y). \tag{4.6}$$

Letting

$$K_1 = A_1 \cdot K \cdot C_1 \quad \text{and} \quad K_2 = \frac{A_2}{K \cdot C_1} + C_2$$

the theorem follows.

■

5. Bounding the core volume from above

Our goal in this section will be to prove the following theorem.

Theorem 5.1. *Given S, there are constants K_3 and K_4 so that if $Q(X,Y) \in QF(S)$ is a quasi-Fuchsian manifold and P_X and P_Y are pants decompositions for which $X \in V(P_X)$ and $Y \in V(P_Y)$ then we have*

$$\mathrm{vol}(X,Y) \leq K_3 d_{\mathbf{P}}(P_X, P_Y) + K_4.$$

Given Theorem 4.1 and Theorem 3.2, Theorem 5.1 represents the final step in the proof of Theorem 1.2.

Let $G \subset \mathbf{P}(S)$ be a shortest path joining P_X and P_Y so that the length of G is simply $d_{\mathbf{P}}(P_X, P_Y)$. Let

$$\mathrm{spin}(G) = \{\alpha^* \mid \alpha \in P,\ P \in G\}$$

denote the geodesic representatives in $Q(X,Y)$ of elements of the pants decompositions along G. We call these geodesics the *spinning geodesics* for G; they will serve to anchor various tetrahedra in $Q(X,Y)$ at their vertices; we will then "spin" these tetrahedra by pulling their vertices around the geodesics.

Our upper bound for $\mathrm{vol}(X,Y)$ will come from a model manifold $N = S \times I$ comprised of blocks that are adapted to $\mathrm{spin}(G)$, together with a piecewise C^1 surjective homotopy equivalence $f\colon N \to \mathcal{N}_\epsilon(C(X,Y))$ so that the image of each block under f has uniformly bounded volume. The model will decompose into two parts.

1. **The Caps:** At each end of N are *caps*, namely products $S \times I$ on which f restricts to homotopies of simplicial hyperbolic surfaces
$$h_X\colon Z_X \to C(X,Y) \quad \text{and} \quad h_Y\colon Z_Y \to C(X,Y)$$
realizing P_X and P_Y to the boundary components X_h^ϵ and Y_h^ϵ of the ϵ-neighborhood $\mathcal{N}_\epsilon(C(X,Y))$ of the convex core.
2. **The Triangulated Part:** The caps sit at either end of the *triangulated part* N_Δ, a union of tetrahedra on which f is simplicial: f lifts to a map sending each simplex to the convex hull of its vertices. It follows that the image of each tetrahedron $\Delta \in N_\Delta$ under f has uniformly bounded volume. We use the geodesics α^*, where $\alpha \in P \in G$, as a scaffolding to

build N_Δ, a glueing of tetrahedra whose image interpolates between the simplicial hyperbolic surfaces Z_X and Z_Y. After "spinning" f sufficiently far about the spinning geodesics, all but a constant times $d_{\mathbf{P}}(P_X, P_Y)$ of the tetrahedra in N_Δ have images with small volume.

These two arguments give the desired bound after collecting constants.

Remark: The above spinning trick is inspired by the ideal simplicial maps of [Th2] which are in effect a limit of the spinning process we perform here. The result in our context of passing to such a limit is an *ideal* triangulation of all but a bounded volume portion of $C(X, Y)$, with a uniformly bounded number of ideal tetrahedra necessary to accomplish each individual elementary move (the small volume tetrahedra collapse to lower dimensional ideal edges and faces). We have chosen to work with finite triangulations in the interest of demonstrating how the combinatorics of $\mathbf{P}(S)$ may be used to produce triangulations of 3-manifolds an semi-algorithmic manner, independent of any geometric structure.

5.1. **Triangulations of surfaces.** We specify a type of triangulation of S that is suited to a pants decomposition P. By a *pair of pants* we will mean a connected component $\widehat{S}$ of $S - \mathcal{N}(P)$, the complement of the union of pairwise disjoint open annular neighborhoods $\mathcal{N}(P)$ of the curves in P on S.

Definition 5.2. *A* standard triangulation $T(\widehat{S})$ *for a pair of pants* $\widehat{S}$ *is a triangulation with the following properties:*

(1) $T(\widehat{S})$ *has two vertices on each boundary component.*
(2) $T(\widehat{S})$ *has two disjoint* spanning triangles *with no vertices in common, and a vertex on each component of* $\partial\widehat{S}$.
(3) *The remaining 3 quadrilaterals are diagonally subdivided by an arc that travels "left to right" with respect to the inward pointing normal to* $\partial\widehat{S}$ *(see Figure 6).*

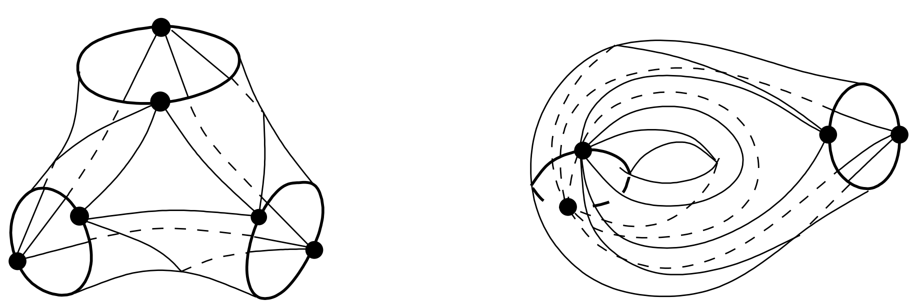

Figure 6. Standard triangulations suited to a pants decomposition.

We construct a standard triangulation suited to a pants decomposition $P \in \mathbf{P}(S)$ by gluing together standard triangulations on pairs of pants $\widehat{S}$ as follows.

Definition 5.3. *Given a pants decomposition* $P \in \mathbf{P}(S)$, *a* standard triangulation suited to P *is a triangulation* T *of* S *obtained as follows (see Figure 6):*

(1) T *has two vertices* p_α *and* $\bar{p}_\alpha$ *on each component* α *of* P, *and two edges* e_α *and* $\bar{e}_\alpha$ *in the complement* $\alpha - p_\alpha \cup \bar{p}_\alpha$.

(2) *If S_0 is a complementary open pair of pants in $S - P$, then the restriction of T to S_0 is the restriction to S_0 of a standard triangulation of $S_0 \cup \partial S_0$ (in the sense of Definition 5.2).*
(3) *If two boundary components α_1 and α_2 of S_0 are identified in S, then the edge of each spanning triangle that runs from α_1 to α_2 forms a closed loop.*

Moves on triangulations. Given an elementary move on pants decompositions (P, P'), i.e. $d_{\mathbf{P}}(P, P') = 1$, we now describe simple moves on triangulations that allow us to move from a standard triangulation suited to P to a standard triangulation suited to P'. To distinguish moves on triangulations from moves on pants decompositions, we refer to the latter as *pants moves*.

To fix notation, given a pants move (P, P') let $\alpha \in P$ and $\beta \in P'$ be the curves for which $i(\alpha, \beta) \neq 0$. We call α and β the curves *involved in the pants move* (P, P'). Let $\mathcal{N}(P - \alpha)$ denote the union of pairwise disjoint open annular neighborhoods about the curves in $P - \alpha$, and let S_α denote the essential subsurface of $S - \mathcal{N}(P - \alpha)$ containing α. If S_α has genus 1 then (P, P') is called a *genus 1* pants move. Likewise, S_α has genus 0 then (P, P') is called a *genus 0* pants move. We say the pants move (P, P') occurs on S_α.

A standard triangulation T suited to P naturally identifies candidate elementary moves for each $\alpha \in P$: there is a natural choice of isotopy class of simple closed curves $\beta \subset S_\alpha$ for which $i(\alpha, \beta) = 1$ or $i(\alpha, \beta) = 2$ depending on whether S_α has genus 1 or genus 0. If S_α has genus 1, then each spanning triangle for T in S_α has one edge with its endpoints identified. These edges are in the same isotopy class which we call $\beta(\alpha, T)$. Likewise, if S_α has genus 0, then removing the edges of T that do not have endpoints lying on α produces two hexagons in the complement of the remaining edges. Concatenating edges in these hexagons joining the two vertices in each that lie on α we obtain an isotopy class of simple closed curves, which we again call $\beta(\alpha, T)$.

For each $\alpha \in P$, the pants decomposition $P' = (P - \alpha) \cup \beta(\alpha, T)$ satisfies $d_{\mathbf{P}}(P, P') = 1$.

There are three basic types of moves on these triangulations:

MVI. *The Dehn twist move.* One standard move on triangulations we will use effects a Dehn twist of a standard triangulation suited to P about a curve $\alpha \in P$. Given a triangulation T suited to P, let $TW_\alpha(T)$ denote the standard triangulation suited to P obtained by shifting each edge with a vertex on α to the right along α until it hits the next vertex. Then $TW_\alpha(T)$ is isotopic to the image of T under a right-α Dehn twist. We define $TW_\alpha^{-1}(T)$ similarly, by shifting edges to the left rather than to the right.

Note that for any triangulation T suited to P and any $\alpha \in P$ we have

$$\beta(\alpha, TW_\alpha(T)) = \tau_\alpha(\beta(\alpha, T))$$

where τ_α is a right α-Dehn twist.

The other elementary moves on triangulations will be specific to a given type of pants move. Given a triangulation T suited to P and $\alpha \in P$, we describe blow-down and blow-up moves that allow us to pass from a triangulation suited to P to a triangulation suited to the pants decomposition $(P - \alpha) \cup \beta(\alpha, T)$.

MVII. *Genus 1 moves.* Given a standard triangulation T suited to P the edges of T that close to form loops in the isotopy class of $\beta(\alpha, T)$ bound an annulus on

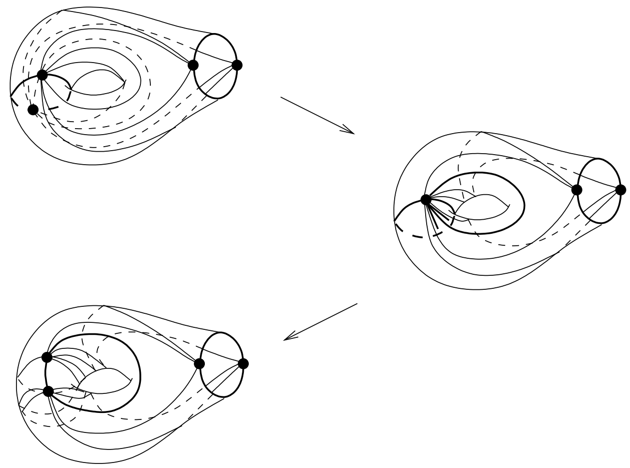

Figure 7. The genus 1 blow-down and blow-up moves.

S. We call this annulus A_α the α *compressing annulus* for T. The *genus 1* α*-blow down* $BD^1_\alpha(T)$ of T is the triangulation of S obtained by collapsing the arc e of $\alpha \cap A_\alpha$ to a point and collapsing the two triangles in T containing e to a single edge.

Let $\beta = \beta(\alpha, T)$. Then given $T' = BD^1_\alpha(T)$ the *genus 1* β*-blow-up* $BU^1_\beta(T')$ of T', is obtained by grafting a compressing annulus A_β in along the curve α to obtain a standard triangulation suited to $(P - \alpha) \cup \beta$ as in Figure 7.

MVIII. *Genus 0 moves.* Let T be a standard triangulation suited to P. Let e_α and $\bar{e}_\alpha$ denote the two edges of T that constitute the curve α. Call these curves the α-edges of T. As described above there are two hexagons H and $\bar{H}$ obtained by removing all edges of T that do not have endpoints on α. The *genus 0* α*-blow-down* $BD^0_\alpha(T) = T'$ (Figure 9) is obtained by performing 3 "diagonal switches" on each hexagon (Figure 8) to yield a new triangulation with edges e_β and $\bar{e}_\beta$ as edges

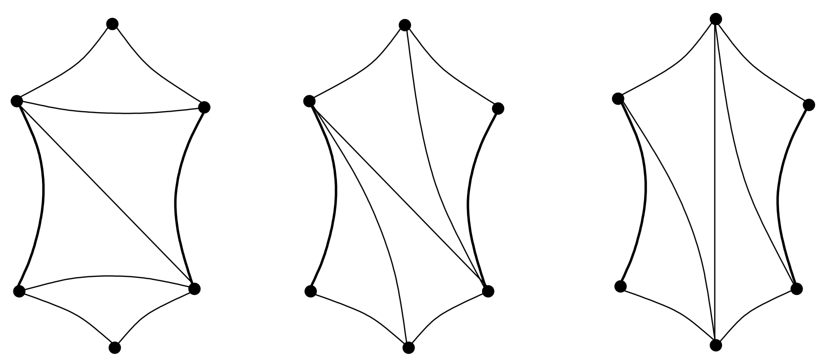

Figure 8. Diagonal switches on hexagons.

whose concatenation gives the curve $\beta = \beta(\alpha, T)$.

Likewise, the *genus 0* β*-blow-up* $BU^1_\beta(T')$ is obtained by modifying the hexagons containing the α-edges by the inverses of the 3-diagonal switches. Note that $BU^0_\beta \circ BD^0_\alpha(T)$ is a standard triangulation suited to the pants decomposition $(P - \alpha) \cup \beta$.

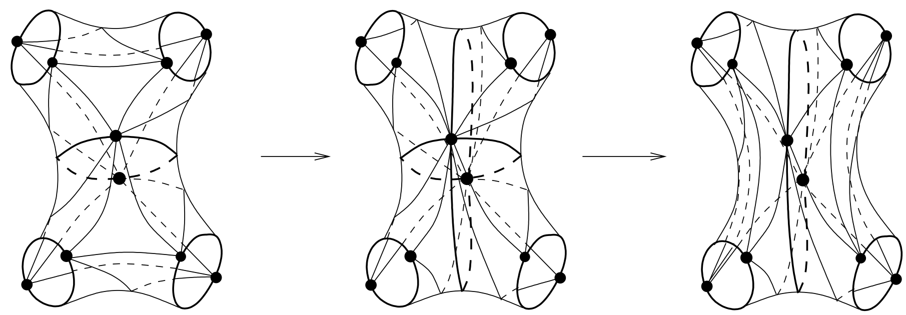

Figure 9. The genus 0 blow-down and blow-up moves.

We summarize properties of these moves as a lemma.

Lemma 5.4. *Let (P, P') be a pants move involving $\alpha \in P$ and $\alpha' \in P'$. Let T be a standard triangulation suited to P. Then there is an $n \in \mathbb{Z}$ so that*

$$BU_{\alpha'} \circ BD_\alpha \circ TW^n_\alpha(T)$$

is a standard triangulation suited to P'.

5.2. **Realizing moves by blocks.** As before let $Q(X, Y)$ be a quasi-Fuchsian manifold, and let P_X and P_Y be pants decompositions for which $X \in V(P_X)$ and $Y \in V(P_Y)$. Let $G \subset \mathbf{P}(S)$ be a geodesic joining P_X and P_Y. Recall we denote by $\mathrm{spin}(G) = \{\alpha^* \mid \alpha \in P,\ P \in G\}$ the *spinning geodesics* associated to G.

Equip each spinning geodesic α^* with a pair of antipodal vertices p_α and $\bar{p}_\alpha$: i.e. points on α^* so that the distance from p_α to $\bar{p}_\alpha$ along α^* is maximal. For reference, we equip each α and thence each α^* with an orientation.

Let P_i, $i = 0, \ldots, m$, denote the pants decompositions along the geodesic G, so that $P_0 = P_X$ and $P_m = P_Y$. Making an initial choice of standard triangulation T_X suited to P_X, Lemma 5.4 provides a sequence of moves on triangulations allowing us to process from the triangulation T_X to a standard triangulation T_Y suited to Y via standard triangulations T_i suited to P_i. We begin with a model manifold

$$N_0 \cong S \times I$$

and triangulate $\partial^+ N_0 = S \times \{0\}$ by T_X. Our aim is to build models $N_i \cong S \times I$ by gluing triangulated I-bundles to $\partial^+ N_{i-1}$ so that the resulting triangulation on $\partial^+ N_i = T_i$. We do this by building a triangulated subsurface block corresponding to each elementary move and successively gluing the blocks to $\partial^+ N_{i-1}$.

Definition 5.5. *Given a curve α in a pants decomposition P, we define the* subsurface block *by the quotient*

$$B_\alpha = S_\alpha \times [0,1]/(x,t) \sim (x,0) \quad \textit{for} \quad x \in \partial S_\alpha, \quad t \in [0,1].$$

We denote the upper *and* lower *boundary of B_α by*

$$\partial^+ B_\alpha = S \times \{0\} \quad \textit{and} \quad \partial^- B_\alpha = S \times \{1\}.$$

We now describe *block triangulations* associated to each elementary move on triangulations. We will say a triangulation $\mathcal{T}$ of a subsurface block B_α *realizes* an elementary move $T \to \mathcal{M}(T)$ if we have

$$\mathcal{T} \cap \partial^- B_\alpha = T \quad \text{and} \quad \mathcal{T} \cap \partial^+ B_\alpha = \mathcal{M}(T).$$

Block triangulations. Let T be a standard triangulation on S_α suited to α. Then the *standard block triangulation* $\mathcal{T}_\alpha$ is obtained from

$$T \times [0,1]/\sim$$

in the following way. Initially, $T \times [0,1]/\sim$ is a cell decomposition of B_α. For any edge e of T with $\partial e \cap \partial S_\alpha = \varnothing$, $e \times [0,1]$ is a quadrilateral to which we add a diagonal depending on the genus of S_α.

- When S_α has genus 0, the α-edges e_α and $\bar{e}_\alpha$ determine quadrilaterals $e_\alpha \times [0,1]$ and $\bar{e}_\alpha \times [0,1]$ in $\mathcal{T}$. We triangulate these quadrilaterals with two new edges that run in the same direction along the annulus

$$e_\alpha \times [0,1] \cup \bar{e}_\alpha \times [0,1]$$

and cone off the new edges down to the vertices opposite the quadrilaterals (see Figure 10).

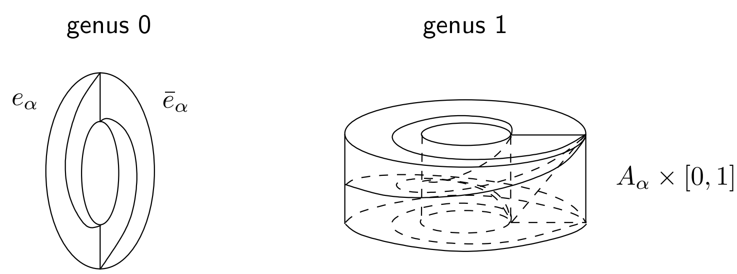

Figure 10. Extending $T \times [0,1]/\sim$ to a triangulation.

- When S_α has genus 1 in addition to the two edges e_α and $\bar{e}_\alpha$ that concatenated give α there are 3 other edges e_1, e_2 and e_3 that triangulate the α-compressing annulus A_α for which $\partial e_j \cap \partial S_\alpha = \varnothing$, $j = 1,2,3$. We triangulate the annulus

$$e_\alpha \times [0,1] \cup \bar{e}_\alpha \times [0,1]$$

as before, and extend this to a triangulation of $A_\alpha \times [0,1]$ with no new vertices as in Figure 10.

From the standard block triangulation, we build four types of blocks:

BLI. *The Dehn twist block.* Given S_α, the block triangulation $\mathcal{T}(TW_\alpha)$ realizing the move TW_α is obtained form the standard block triangulation $\mathcal{T}_\alpha$ as follows. Consider the annulus $A = \alpha \times [0,1]$ in B_α with the triangulation T_A on A induced by $\mathcal{T}_\alpha$. The reference orientation for α locally determines a *left* and *right* side of α in S_α, and hence a left and right side of A in B_α. Cut B_α along A to obtain two annuli A_L and A_R that bound the *local* left and right side of $B_\alpha - A$. Re-glue the $\alpha \times \{0\}$ boundary components of A_L and A_R by the identity, and re-glue the $\alpha \times \{1\}$ boundary components of A_L and A_R shifted by a right Dehn twist.

The triangulations induced by T_A on A_L and A_R determine a triangulation of the torus $A_L \cup A_R$ after re-gluing. This triangulation naturally extends to a triangulation of a solid torus V with boundary $A_L \cup A_R$ by filling in tetrahedra (the triangulations on A_L and A_R differ by two pairs of diagonal switches. See Figure 11). After filling in by V, the result is a standard block B_α with triangulation $\mathcal{T}_\alpha^{\text{tw}+}$ realizing the move TW_α. We call the standard block B_α equipped with the triangulation $\mathcal{T}_\alpha^{\text{tw}+}$ the *right Dehn twist block* $B_\alpha^{\text{tw}+}$. The *left Dehn twist block*

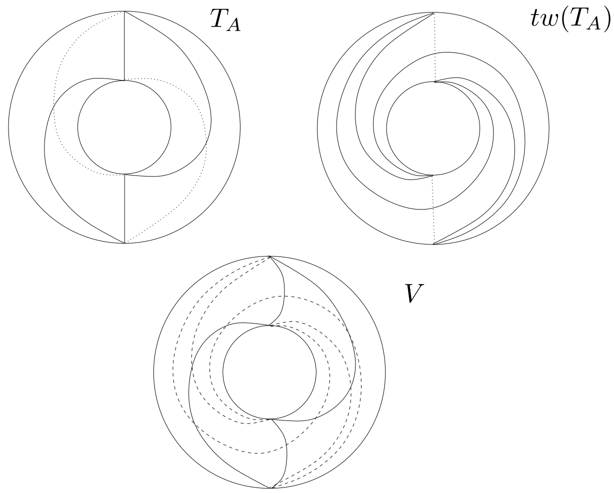

Figure 11. Triangulating the Dehn twist block.
The dotted edges indicate diagonal switches for T_A and $tw(T_A)$ that produce a common triangulation of the annulus, which results in the pictured triangulation of a solid torus V.

$B_\alpha^{\text{tw}-}$ is obtained analogously, by re-gluing with a left Dehn twist instead of a right Dehn twist. The left Dehn twist block $B_\alpha^{\text{tw}-}$ carries the triangulation $\mathcal{T}_\alpha^{\text{tw}-}$ and realizes the move TW_α^{-1}.

Blocks realizing the other moves are triangulated as follows.

BLII. *Blow-up and blow-down blocks.* We modify the standard block triangulation to obtain triangulated blocks that realize blow-up and blow-down moves as follows. To realize a genus 1 α-blow-down by a block triangulation, we modify the standard block triangulation $\mathcal{T}_\alpha$ on B_α by collapsing the compression annulus on $\partial^+ B_\alpha$ as in the description of the move BD_α^1. The only difference is that here in addition to collapsing triangles to edges we also collapse tetrahedra to triangles.

Similarly, if $\beta = \beta(\alpha, T)$, we obtain a block triangulation realizing the genus 1 β-blow-up BU_β^1 by collapsing the β compression annulus on $\partial^- B_\beta$.

The genus 0 α-blow-down (and blow-up) moves come from diagonal switches on hexagons. We realize the move BD_α^0 by gluing tetrahedra realizing each of the diagonal switches to $\partial^+ B_\alpha$. Likewise, the block triangulation realizing the genus 0 β-blow-up can be obtained by gluing tetrahedra in this way to the β-hexagons on $\partial^- B_\alpha$.

We denote the α-blow-up block by B_α^{bu} and the α-blow-down block by B_α^{bd}.

BLIII. *Straightening blocks.* There are two other types of block triangulations we will need that realize the identity move on a blown-down triangulation. We call these triangulated blocks *straightening blocks*. Given the triangulation $BD^1_\alpha(T)$ or $BD^0_\alpha(T)$ for a standard triangulation T suited to a pants decomposition P containing α, the genus g straightening triangulation, $g = 0, 1$ is obtained by completing the cell decomposition

$$(BD^g_\alpha(T) \cap S_\alpha) \times [0,1]/\sim$$

of B_α, and extending this decomposition to a triangulation of B_α with no new vertices. It is easy to check that this can be done. We denote the straightening blocks by $B^{\mathrm{st}}_{\alpha,\beta}$ where $\beta = \beta(\alpha, T)$.

5.3. **Mapping in blocks and building the model.** We now use our block triangulations to build a *model manifold* $N \cong S \times I$. We build N in stages corresponding to standard triangulations T_j suited to pants decompositions P_j that intervene between P_X and P_Y respectively. At each stage there is a model N_j also homeomorphic to $S \times I$ so that the top boundary component $\partial^+ N_j = S \times \{1\}$ is triangulated by the jth triangulation T_j in the sequence of triangulations. Each N_j will be obtained from N_{j-1} by attaching a triangulated block to $\partial^+ N_{j-1}$ that realizes the elementary move needed to move from T_{j-1} to T_j.

The model N will come equipped with a map $f\colon N \to C(X,Y)$ that is *simplicial* on each block:

Definition 5.6. *An incompressible mapping $f\colon N \to M$ from a triangulated 3-manifold to a hyperbolic* 3*-manifold is* simplicial *if the lift $\widetilde{f}\colon \widetilde{N} \to \mathbb{H}^3$ sends each k-simplex to the convex hull of its vertices.*

The following theorem describes the properties of our model and its mapping to the quasi-Fuchsian manifold $Q(X,Y)$. For simplicity we assume for the remainder of this section that S is closed, and detail the necessary modifications to the argument at the end.

Theorem 5.7. *Given $Q(X,Y)$, there is a model manifold $N \cong S \times I$, equipped with a surjective homotopy equivalence*

$$f\colon N \to \mathcal{N}_\epsilon(C(X,Y))$$

to the ϵ-neighborhood of the convex core $C(X,Y)$ with the following properties:

(1) *N is the union*

$$N = \mathrm{cap}_X \cup N_\Delta \cup \mathrm{cap}_Y$$

of the caps $\mathrm{cap}_X \cong S \times I$ *and* $\mathrm{cap}_Y \cong S \times I$ *and the* triangulated part N_Δ, *a union of blocks of the above type glued top boundary to bottom boundary, all but $3d_{\mathbf{P}}(P_X, P_Y)$ of which are* Dehn twist *blocks.*

(2) *The map f is piecewise C^1 and is simplicial on N_Δ.*

(3) *For each tetrahedron Δ in N_Δ that lies in a Dehn twist block, f maps some edge of Δ to a spinning geodesic.*

(4) *The restriction $f|_{\mathrm{cap}_X}$ is a homotopy from $\partial^- \mathcal{N}_\epsilon(C(X,Y))$ to a simplicial hyperbolic surface realizing P_X, and the restriction $f|_{\mathrm{cap}_Y}$ is a homotopy from a simplicial hyperbolic surface realizing P_Y to $\partial^+ \mathcal{N}_\epsilon(C(X,Y))$.*

Proof: To motivate the construction of N we build the map f in stages as well.

Mapping in cap_X. Let T_X and T_Y be standard triangulations suited to P_X and P_Y. Let

$$f_X\colon S \times [0,1] \to C(X,Y)$$

determine a homotopy of the convex core boundary $g_X : X_h \to \partial^- C(X,Y)$ to a simplicial hyperbolic surface $h_X : Z_X \to C(X,Y)$ with associated triangulation T_X so that if v and $\bar{v}$ are the vertices of T_X on $\alpha \in P_X$, $f_X(v) = p_\alpha$ and $f_X(\bar{v}) = \bar{p}_\alpha$. Notice that this implies that h_X realizes each curve $\alpha \in P_X$. Let $N_0 = S \times [0,1]$, denote the domain for f_X; we will refer to $N_0 = \text{cap}_X$ as the X-cap of N. The top boundary component $\partial^+ N_0$ carries the triangulation T_X.

Working inductively, we assume given a model N_j at the jth stage: i.e.

(1) the model $N_j \cong S \times I$, consists of the X-cap and a triangulated part so that the upper boundary $\partial^+ N_j$ is triangulated by a standard triangulation T_j suited to P_j,
(2) N_j comes equipped with a map $f_j : N_j \to C(X,Y)$ that is simplicial on the triangulated part of N_j,
(3) $f_j|_{\partial^+ N_j}$ factors through a simplicial hyperbolic surface $h_j : Z_j \to C(X,Y)$ with associated triangulation T_j so that h_j sends vertices v_γ and $\bar{v}_\gamma$ on $\gamma \in P_j$ to p_γ and $\bar{p}_\gamma$ on γ^*.

Let $\alpha \in P_j$ and $\alpha' \in P_{j+1}$ be the curves involved in the genus g elementary move (P_j, P_{j+1}), and let n be the integer guaranteed by Lemma 5.4 for which

$$BU^g_{\alpha'} \circ BD^g_\alpha \circ TW^n_\alpha(T_j) = T_{j+1}$$

is a standard triangulation suited to P_{j+1}.

We now specify how to add triangulated blocks to N_j and extend f_j simplicially over each additional block to obtain the next stage of the model N_{j+1}.

Mapping in Dehn twist blocks. If n is positive, we attach n right Dehn twist blocks to the model and extend f_j over them in sequence, while if n is negative we do likewise with left Dehn twist blocks.

We assume n is positive; the negative case is identical with left Dehn twist blocks replacing right Dehn twist blocks. We attach a right Dehn twist block $B^{\text{tw+}}_\alpha$ to $\partial^+ N_j$ so that the triangulation on $\partial^- B^{\text{tw+}}_\alpha$ agrees with $T_j \cap S_\alpha$ to obtain a new model $N_{j,1}$. We extend f_j over $B^{\text{tw+}}_\alpha$ to obtain a map $f_{j,1}$ as follows. Recalling that $B^{\text{tw+}}_\alpha$ has the form $S_\alpha \times [0,1]/\sim$, we set $f_j(x,t) = f_j(x,0)$. We then straighten f_j on $\alpha \times \{1\}$ to its geodesic representative so that the vertices v and $\bar{v}$ on $\alpha \times \{1\}$ to p_α and $\bar{p}_\alpha$, and finally we straighten $f_{j,1}$ by a homotopy to make it simplicial on $B^{\text{tw+}}_\alpha$. We note that every tetrahedron in the Dehn twist block has an edge that maps to a geodesic arc of α^*, verifying part (3).

The map $f_{j,1}$ factors through a simplicial hyperbolic surface still realizing P_j with associated triangulation $TW_\alpha(T_j)$ which we denote by $T_{j,1}$. Repeating this procedure to add n Dehn twist blocks we arrive at a model $N_{j,n}$ equipped with a map $f_{j,n} : N_{j,n} \to C(X,Y)$ so that

(1) $\partial^+ N_{j,n}$ carries the triangulation

$$T_{j,n} = TW^n_\alpha(T_j),$$

(2) $f_{j,n}$ factors through a simplicial hyperbolic surface realizing P_j with associated triangulation $T_{j,n}$, and
(3) the vertices of $T_{j,n}$ map to p_γ and $\bar{p}_\gamma$ on γ^* for each $\gamma \in P_j$.

Mapping in blow-down blocks. Our discussion of how to attach a blow-down block to $N_{j,n}$ and how to extend $f_{j,n}$ over this block breaks into cases as usual.

Genus 0. The genus 0 blow-down block B_α^{bd} is attached to $\partial^+ N_{j,n}$ along S_α so that the triangulations agree as before. This gives a new model $N_{j,\mathrm{bd}}$. We extend $f_{j,n}$ over B_α^{bd} to give a map $f_{j,\mathrm{bd}} \colon N_{j,\mathrm{bd}} \to C(X,Y)$ by first mapping B_α^{bd} to $\partial^+ N_{j,n}$, as with the Dehn twist block, and then straightening $f_{j,\mathrm{bd}}$ to a simplicial map.

Genus 1. In the genus 1 case is the same, except that as there is only one vertex v on α in $\partial^+ B_\alpha^{\mathrm{bd}}$, we simply send v to p_α and straighten $f_{j,\mathrm{bd}}$ to a simplicial map as before.

In each case the resulting map $f_{j,\mathrm{bd}}|_{\partial^+ N_{j,\mathrm{bd}}}$ factors through a simplicial hyperbolic surface that realizes P_j and has associated triangulation $BD_\alpha^0(T_{j,n})$ which we denote by $T_{j,\mathrm{bd}}$.

Mapping in straightening blocks. Straightening blocks allow us to pass from simplicial hyperbolic surfaces realizing P_j to simplicial hyperbolic surfaces realizing P_{j+1}. We attach the straightening block $B^{\mathrm{st}}_{\alpha,\alpha'}$ to $N_{j,\mathrm{bd}}$ to obtain a model $N_{j,\mathrm{st}}$. We extend $f_{j,\mathrm{bd}}$ over $B^{\mathrm{st}}_{\alpha,\alpha'}$ to a map $f_{j,\mathrm{st}}$ by defining $f_{j,\mathrm{st}}(x,t) = f_{j,\mathrm{st}}(x,0)$, and then straightening $f_{j,\mathrm{st}}$ on $\alpha' \times \{1\} \subset \partial^+ B^{\mathrm{st}}_{\alpha,\alpha'}$. We send the α' vertex to $p_{\alpha'}$ or vertices to $p_{\alpha'}$ and $\bar{p}_{\alpha'}$ and straighten the map to a simplicial map on $B^{\mathrm{st}}_{\alpha,\alpha'}$. Now

$$f_{j,\mathrm{st}}|_{\partial^+ N_{j,\mathrm{st}}}$$

factors through a simplicial hyperbolic surface realizing P_{j+1} with associated triangulation $T_{j,\mathrm{bd}}$ once again.

Mapping in blow-up blocks. This procedure is essentially the inverse of the attaching and mapping in the blow-down blocks.

Genus 0. Attach $B^{\mathrm{bu}}_{\alpha'}$ to $\partial^+ N_{j,\mathrm{st}}$ along $S_{\alpha'}$ (which equals S_α), and extend $f_{j,\mathrm{st}}$ to $f_{j,\mathrm{bu}}$ over $B^{\mathrm{bu}}_{\alpha'}$ by setting $f_{j,\mathrm{bu}}(x,t) = f_{j,\mathrm{bu}}(x,0)$ and then straightening $f_{j,\mathrm{bu}}$ *rel*-$\alpha' \times \{1\}$ to a simplicial map.

Genus 1. Set $f_{j,\mathrm{bu}}(x,t) = f_{j,\mathrm{bu}}(x,0)$ and then homotope $f_{j,\mathrm{bu}}$ so that it still sends $\alpha' \subset \partial^+ B^{\mathrm{bu}}_{\alpha'}$ to its geodesic representative but also sends the α' vertices $v_{\alpha'}$ and $\bar{v}_{\alpha'}$ to $p_{\alpha'}$ and $\bar{p}_{\alpha'}$.

The map $f_{j,\mathrm{bu}}|_{\partial^+ N_{j,\mathrm{bu}}}$ factors through a simplicial hyperbolic surface realizing P' with associated triangulation

$$T_{j+1} = BU_{\alpha'} \circ BD_\alpha \circ TW_\alpha^n(T_j)$$

which is a standard triangulation suited to P_{j+1}. This completes the inductive step.

Mapping in cap_Y. Let $|G| = d_{\mathbf{P}}(P_X, P_Y)$ denote the length of a geodesic $G \subset \mathbf{P}(S)$. Then the above inductive procedure results finally in a map

$$f_{|G|} \colon N_{|G|} \to C(X,Y)$$

so that the restriction $f_{|G|,0}|_{\partial^+ N_{|G|}}$ factors through a simplicial hyperbolic surface realizing P_Y with associated triangulation T_Y (a standard triangulation suited to P_Y), and so that the vertices v_γ and $\bar{v}_\gamma$ map to the vertices p_γ and $\bar{p}_\gamma$ on the closed geodesic γ^* for each $\gamma \in P_Y$. We complete our model N by adding a Y-cap: this is a homotopy

$$f_Y \colon S \times I \to C(X,Y)$$

from $f_{|G|}|_{\partial^+ N_{|G|}}$ to the convex core boundary $g_Y\colon Y_h \to \partial^+ C(X,Y)$. Gluing this homotopy $S \times I$, to $\partial^+ N_{|G|}$ and extending $f_{|G|}$ over the Y-cap by f_Y, we obtain the final piece of our model N and the resulting map

$$f\colon N \to C(X,Y),$$

a homotopy equivalence whose restrictions

$$f|_{\partial^+ N} \to \partial^+ C(X,Y) \quad \text{and} \quad f|_{\partial^- N} \to \partial^- C(X,Y)$$

are homeomorphisms.

Though the boundary $\partial C(X,Y)$ is not generically smooth, by taking the boundary $\partial \mathcal{N}_\epsilon(C(X,Y))$ of the ϵ-neighborhood of the convex core we obtain a pair of C^1 surfaces $\partial^+\mathcal{N}_\epsilon(C(X,Y))$ and $\partial^-\mathcal{N}_\epsilon(C(X,Y))$ with C^1 path metrics [EM, Lem. 1.3.6]. In the interest of computing volume, we perturb f to a piecewise smooth map

$$f^\epsilon\colon N \to \mathcal{N}_\epsilon(C(X,Y))$$

by adjusting f by a homotopy that changes f only on cap_X and cap_Y, so that

$$f^\epsilon|_{\partial^+ N}\colon \partial^+ N \to \partial^+\mathcal{N}_\epsilon(C(X,Y)) \quad \text{and} \quad f^\epsilon|_{\partial^- N}\colon \partial^- N \to \partial^-\mathcal{N}_\epsilon(C(X,Y))$$

are homeomorphisms, and f^ϵ is C^1 on the interiors of the caps of N. The map f is already simplicial on the triangulated part of N so f^ϵ is piecewise C^1. A degree argument shows f^ϵ is surjectve, proving the theorem.

■

5.4. **Bounding the volume.** Given a piecewise differentiable 3-chain C in a hyperbolic 3-manifold M, the function $\deg_C\colon M \to \mathbb{Z}$ which measures the degree of C in M is well defined at almost every point of M. We define the *mass* $\mathrm{mass}(C)$ of C to be the integral

$$\mathrm{mass}(C) = \int_M |\deg_C| dV$$

where dV is the hyperbolic volume form on M (cf. [Th2, §4]).

Moreover, if $F\colon P \to M$ is a map of a piecewise differentiable 3-manifold P to M, and C is a piecewise differentiable 3-chain in P, then we define the F-*mass* of C by the integral

$$\mathrm{mass}_F(C) = \int_M |\deg_{F(C)}| dV.$$

The F-mass of C bounds the volume $\mathrm{vol}(F(C))$ of the image of C in M. Hence, given our piecewise differentiable surjective map

$$f^\epsilon\colon N \to \mathcal{N}_\epsilon(C(X,Y))$$

the volume $\mathrm{vol}(\mathcal{N}_\epsilon(C(X,Y))$, which bounds $\mathrm{vol}(X,Y)$, is bounded by the sums of the f^ϵ-masses of the chains that make up N. In other words, if N decomposes into 3-chains C_k, we have

$$\mathrm{vol}(X,Y) \le \sum_{C_k \subset N} \mathrm{mass}_{f^\epsilon}(C_k) =: \mathrm{mass}_{f^\epsilon}(N).$$

Thus, Theorem 5.1 will follow from the following proposition.

Proposition 5.8. *Given S there are constants $K_3 > 1$ and $K_4 > 0$ so that the map f^ϵ is properly homotopic to a map $f^\epsilon_\theta \colon N \to \mathcal{N}_\epsilon(C(X,Y))$ for which*

$$\mathrm{mass}_{f^\epsilon_\theta}(N) \leq K_3 d_{\mathbf{P}}(P_X, P_Y) + K_4.$$

Let $\mathcal{V}_3$ denote the maximal volume of a tetrahedron in hyperbolic 3-space (see [Th1, ch. 7], [BP]). We begin our approach to Proposition 5.8 with the following lemma.

Lemma 5.9. *There is a constant K_Δ so that the map f^ϵ is properly homotopic to a map f^ϵ_θ that also satisfies the conclusions of Theorem 5.7 so that*

$$\mathrm{mass}_{f^\epsilon_\theta}(N_\Delta) < K_\Delta \cdot \mathcal{V}_3 \cdot d_{\mathbf{P}}(P_X, P_Y) + 1.$$

Proof: Because of the possibility of a large number of Dehn twist blocks in N_Δ, there is not in general a uniform constant K for which the number of tetrahedra used to triangulate N_Δ is less than $K d_{\mathbf{P}}(P_X, P_Y)$. We will show, however, that by modifying f^ϵ by a homotopy, we can force the tetrahedra that lie in Dehn twist blocks to have mass as small as we like.

By Theorem 5.7, the number of blocks in N_Δ that are not Dehn twist blocks is bounded by $3d_{\mathbf{P}}(P_X, P_Y)$. Since there is a uniform bound to the number of tetrahedra in any block, there is a constant K_Δ so that all but at most $K_\Delta d_{\mathbf{P}}(P_X, P_Y)$ tetrahedra of N_Δ lie in Dehn twist blocks.

Let $\alpha^* \in \mathrm{spin}(G)$ be a spinning geodesic. Let f^ϵ_θ be defined by the following homotopy of f^ϵ through maps that are simplicial on N_Δ: for each $\alpha \in \cup_j P_j$, slide the vertices p_α and $\bar{p}_\alpha$ along the geodesic α^* a distance

$$\frac{\theta}{2\pi} \ell_{Q(X,Y)}(\alpha^*)$$

in the direction of the reference orientation chosen for α. (See [Th2] for another example of this *spinning* of triangulations). The following lemma shows that tetrahedra that lie in a Dehn twist block, which we will call *Dehn twist tetrahedra*, can be made to have small f^ϵ_θ-mass by spinning to sufficiently high values of θ.

Lemma 5.10. *If $\Delta \subset N_\Delta$ is a Dehn twist tetrahedron, then*

$$\mathrm{mass}_{f^\epsilon_\theta}(\Delta) \to 0 \quad \textit{as} \quad \theta \to \infty.$$

Proof: Recall from Theorem 5.7, each Dehn twist tetrahedron Δ has at least one edge e for which $f^\epsilon(e) \subset \alpha^*$ for some spinning geodesic α^*.

Lift f^ϵ_θ to $\widetilde{f}^\epsilon_\theta \colon \widetilde{N} \to \widetilde{\mathbb{H}^3}$ and choose a lift $\widetilde{\Delta}$ to $\widetilde{N}$. Let $\widetilde{\alpha^*}$ denote the lift of α^* to $\mathbb{H}^3$ for which $\widetilde{f}^\epsilon_\theta$ sends the lifted edge $\widetilde{e} \subset \widetilde{\Delta}$ to $\widetilde{\alpha^*}$. Let e' be the opposite edge of Δ (e and e' have no endpoints in common).

Let I_θ be the ideal tetrahedron in $\mathbb{H}^3$ for which

1. (1) $\widetilde{f}^\epsilon_\theta(\widetilde{e'})$ lies in one edge e'_∞ of I_θ,
2. (2) the two other edges e^1_∞ and e^2_∞ of I_θ emanating from one endpoint of e'_∞ pass through endpoints p_1 and p_2 of $\widetilde{f}^\epsilon_\theta(\widetilde{e})$.

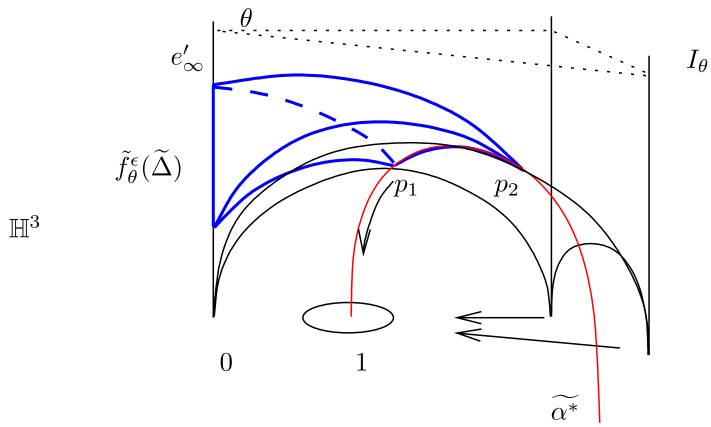

Figure 12. Spinning tetrahedra in $\mathbb{H}^3$.

The image $\widetilde{f}^\epsilon_\theta(\widetilde{\Delta})$ lies in I_θ (see Figure 12).

Normalize by an isometry of $\mathbb{H}^3$ so that the edge e'_∞ of I_θ has its ideal endpoints at 0 and ∞, and so that the terminal fixed point of $\widetilde{\alpha^*}$ (with respect to the reference orientation for α) lies at $1 \in \mathbb{C}$.

Then for any $r > 0$ there exists a θ so that the two ideal vertices of I_θ that do not lie at 0 and ∞ lie within a small disk $|1 - z| < r$. It follows that the dihedral angle of I_θ at e'_∞ tends to 0 as $\theta \to \infty$. But the volume of an ideal tetrahedron tends to 0 as any of its dihedral angles tends to 0, so we have

$$\mathrm{mass}_{f^\epsilon_\theta}(\Delta) < \mathrm{vol}(I_\theta) \to 0 \quad \text{as} \quad \theta \to \infty.$$

∎

Continuation of the proof of Lemma 5.9: By Lemma 5.10, for any Dehn twist block B, the quantity $\mathrm{mass}_{f^\epsilon_\theta}(B)$ is as small as we like for θ sufficiently large. If $\mathcal{B}^{\mathrm{tw}}$ denotes the union of all Dehn twist blocks in N_Δ, then, we may choose θ sufficiently large so that

$$\mathrm{mass}_{f^\epsilon_\theta}(\mathcal{B}^{\mathrm{tw}}) < 1.$$

Since f^ϵ_θ is simplicial on N_Δ, we have

$$\mathrm{mass}_{f^\epsilon_\theta}(\Delta) < \mathcal{V}_3$$

for any tetrahedron $\Delta \subset N_\Delta$, and since all but at most $K_\Delta d_{\mathbf{P}}(P_X, P_Y)$ tetrahedra in N_Δ lie in $\mathcal{B}^{\mathrm{tw}}$, we have

$$\mathrm{mass}_{f^\epsilon_\theta}(N_\Delta) < K_\Delta \cdot \mathcal{V}_3 \cdot d_{\mathbf{P}}(P_X, P_Y) + 1.$$

∎

Bounding the volume of the caps. The bound on the f^ϵ_θ-mass of the triangulated part N_Δ in terms of the distance $d_{\mathbf{P}}(P_X, P_Y)$ will be sufficient for Proposition 5.8 once we show the following uniform bound on the f^ϵ_θ-mass of the caps of N.

Lemma 5.11. *There is a uniform constant K_{cap}, depending only on S so that after modifying f^ϵ_θ by a homotopy on cap_X we have*

$$\mathrm{mass}_{f^\epsilon_\theta}(\mathrm{cap}_X) < K_{\mathrm{cap}}$$

and similarly for cap_Y.

Proof: Our goal will be to modify the homotopy $f^\epsilon_\theta|_{\mathrm{cap}_X}$ from $\partial^- \mathcal{N}_\epsilon(C(X,Y))$ to the simplicial hyperbolic surface Z_X by cutting the surface S into annuli and controlling the trace of the homotopy on each annulus (a solid torus). To obtain such control, we choose this decomposition compatibly with the pants decomposition P_X.

We fix attention on a single pair of pants $\widehat{S} \subset S - P_X$. By a *figure-8 curve* on $\widehat{S}$ we will mean a closed curve that intersects itself once on $\widehat{S}$ and divides $\widehat{S}$ into three annuli, one parallel to each boundary component of $\widehat{S}$ (see Figure 13). To prove Lemma 5.11 we establish the following basic lemma in hyperbolic surface geometry. (We continue to treat the closed case; the case when S has boundary is similar).

Lemma 5.12. *Let S a closed surface with negative Euler characteristic, and let $L' \geq L$ be a constant greater than or equal to the Bers constant L for S. Then there is a constant $L_8(L') > 0$ so that the following holds: if P is a pants decomposition of S, $Z \in \mathrm{Sing}_k(S)$ is a possibly singular hyperbolic surface, and the length bound $\ell_Z(\alpha) < L'$ holds for each $\alpha \in P$, then any figure-8 curve γ in any component $\widehat{S} \subset S - P$ satisfies*

$$\ell_Z(\gamma) < L_8(L').$$

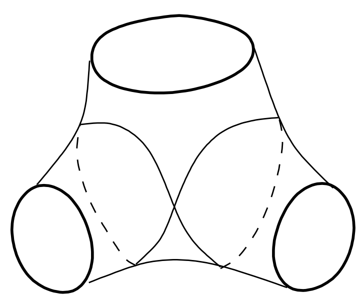

Figure 13. A figure eight curve.

Proof: Note that any bound on the geodesic length of a given figure-8 curve γ guarantees a bound on the geodesic length of any other, by taking twice the original bound.

Assume first that $Z \in \mathrm{Teich}(S)$, so Z is a non-singular hyperbolic surface. From the thick-thin decomposition for hyperbolic surfaces, given any $\delta > 0$, there is a uniform bound D_δ to the diameter $\mathrm{diam}(Z_{\geq\delta})$ of the δ-thick part of Z, where D_δ depends only on δ and the surface S. Let $\partial\widehat{S} = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3$. Since $\ell_Z(\alpha_i) < L'$, there is a δ so that α_i^* can only intersect $Z_{<\delta}$ in a component of $Z_{<\delta}$ for which it is the core curve. Choosing δ smaller if necessary, we can ensure that the boundary of any component of $Z_{<\delta}$ is a curve of length less than L'.

Let $\widehat{Z}$ be the realization of $\widehat{S}$ as a subsurface of Z bounded by the curves $\cup\alpha_i^*$. Either two of the α_i^* lie entirely in $Z_{\geq\delta}$ or two of the α_i are homotopic into $Z_{<\delta}$. Without loss of generality, assume α_1^* and α_2^* lie in $Z_{\geq\delta}$. Then they can be joined by an arc b in $\widehat{Z}$ of length less than D_δ. Either $\alpha_1 \cdot b \cdot \alpha_2 \cdot b^{-1}$ or $\alpha_1 \cdot b \cdot \alpha_2^{-1} \cdot b^{-1}$ is a figure-8 curve of length less than $2L' + 2D_\delta$. Otherwise, if A_1 and A_2 are components of $Z_{<\delta}$ representing the homotopy classes of α_1 and α_2, then there are components a_1 of ∂A_1 and a_2 of ∂A_2 with length less than L', and an arc b joining a_1 to a_2 in $\widehat{Z}$ of length less than D_δ. By the same reasoning, there is a figure-8 curve γ of length

$$\ell_Z(\gamma) < 2L' + 2D_\delta.$$

To treat the potentially singular case, let $Z \in \mathrm{Sing}_k(S)$ satisfy $\ell_Z(\alpha_i) < L'$ for each $\alpha_i \in P$. Let $Z^{\mathrm{h}} \in \mathrm{Teich}(S)$ represent the hyperbolic surface in the same conformal class as Z. If each $\alpha_i \in P$ satisfies

$$\ell_{Z^{\mathrm{h}}}(\alpha_i) < L'$$

then there is a figure-8 curve $\gamma \subset \widehat{S}$ with $\ell_{Z^{\mathrm{h}}}(\gamma) < 2L' + 2D_\delta$ by the above reasoning. By Ahlfors' lemma, [Ah] we have

$$\ell_Z(\gamma) < \ell_{Z^{\mathrm{h}}}(\gamma) < 2L' + 2D_\delta$$

proving the lemma in this case.

If, however, some $\alpha_i \subset \partial\widehat{S}$ satisfies $\ell_{Z^{\mathrm{h}}}(\alpha_i) \geq L' \geq L$, then by Theorem 2.1 there is a simple closed curve β, with $i(\alpha_i, \beta) \neq 0$, for which $\ell_{Z^{\mathrm{h}}}(\beta) < L$. Again applying Ahlfors' lemma, the curve β has length

$$\ell_Z(\beta) < L$$

on Z, so its geodesic representative β^* furnishes an arc b in $\widehat{S}$ that joins α_i^* to α_j^*, or joins α_i^* to itself. In either case, two copies of b together with arcs in the geodesics at its endpoints can be assembled to form a figure-8 curve $\beta \subset \widehat{S}$ with length bounded by

$$\ell_Z(\beta) < 2L' + 2L \leq 4L'.$$

By setting

$$L_8(L') = 4L' + 2D_\delta$$

the lemma follows.

■

Remark: Though we continue to work in the setting where S is closed, the proof of the Lemma for the case when $\partial S \neq \varnothing$ goes through simply.

Continuation of the proof of Lemma 5.11: Again consider the subsurface $\widehat{S}$ and its realization as a subsurface $\widehat{Z_X} \subset Z_X$ bounded by geodesics $\hat{\alpha_1}$, $\hat{\alpha_2}$ and $\hat{\alpha_3}$ on Z_X. Let

$$X_h^\epsilon = \partial^+ \mathcal{N}_\epsilon(C(X,Y))$$

denote the boundary component of $\mathcal{N}_\epsilon(C(X,Y))$ facing X, and let $\widehat{X_h^\epsilon}$ denote the realization of $\widehat{S}$ as a subsurface X_h^ϵ bounded by geodesics $\bar{\alpha_1}$, $\bar{\alpha_2}$ and $\bar{\alpha_3}$ in the path metric on X_h^ϵ. Let γ be a figure-8 curve on $\widehat{S}$ with geodesic representatives $\hat{\gamma}$ and $\bar{\gamma}$ on $\widehat{Z_X}$ and $\widehat{X_h^\epsilon}$ respectively.

Let A_1, A_2 and A_3 denote the three annuli in $\widehat{S} - \gamma$ for which $\alpha_i \subset \partial A_i$, where $i = 1, 2, 3$, and let $\gamma_i \subset \gamma$ be the loop on γ so that $\gamma_i \subset \partial A_i$ for $i = 1, 2, 3$. Let $\hat{A}_i$

be the realization of A_i on $\widehat{Z_X}$ with $\partial \hat{A}_i \subset \hat{\alpha}_i \cup \hat{\gamma}$, and let $\bar{A}_i$ be the realization of A_i on $\widehat{X_h^\epsilon}$ with $\partial \bar{A}_i \subset \bar{\alpha}_i \cup \bar{\gamma}$.

By a theorem of D. Sullivan [Sul1] elaborated upon by D. Epstein and A. Marden [EM, Thm. 2.3.1] the path metric on the surface X_h^ϵ has bounded distortion from the hyperbolic metric on X: the *nearest point retraction* map (see [EM]) is $4\cosh(\epsilon)$-Lipschitz, in particular. It follows that there is a constant $C_5 > 4$ so that choosing ϵ sufficiently small we have

$$\ell_{X_h^\epsilon}(\bar{\alpha}_i) < C_5 L$$

and

$$\operatorname{area}(\widehat{X_h^\epsilon}) < \operatorname{area}(X_h^\epsilon) < C_5 2\pi|\chi(S)|$$

where $i = 1, 2, 3$. Since $\widehat{Z_X}$ is triangulated by 8 hyperbolic triangles, we have

$$\operatorname{area}(\hat{A}_i) < 8\pi$$

for $i = 1, 2, 3$.

Returning to the map $f_\theta^\epsilon \colon N \to \mathcal{N}_\epsilon(C(X,Y))$, we may define the f_θ^ϵ-mass of a piecewise differentiable 2-cycle in N similarly to that of a 3-chain, by integrating the absolute value of the degree with respect to two dimensional Lebesgue-measure on M (cf. [Th2, Prop. 4.1] and the preceding discussion).

We modify the map f_θ^ϵ on $\operatorname{cap}_X \cong S \times I$ as follows. By Theorem 5.7, the map f_θ^ϵ already sends $\alpha_i \times 0$ to the geodesic $\hat{\alpha}_i$ for each $\alpha_i \in P_X$. Straighten f_θ^ϵ without changing its values on $\partial^+\operatorname{cap}_X$ so that f_θ^ϵ sends each $\alpha_i \times \{0\}$ to the geodesic $\bar{\alpha}_i \subset X_h^\epsilon$, and so that f_θ^ϵ sends each track $x \times [0,1]$, where x lies in α_i to a geodesic. The image $f_\theta^\epsilon(\alpha_i \times [0,1])$ is a ruled annulus, and any ruled annulus has area less than the sum of the lengths of its boundary components (see e.g. [Th1, Ch. 9] [Bon, §3.2]). Since $\ell_{X_h^\epsilon}(\bar{\alpha}_i) < C_5 L$, and since $\hat{\alpha}_i$ is the geodesic representative of α_i in $Q(X,Y)$, we have

$$\operatorname{mass}_{f_\theta^\epsilon}(\alpha_i \times [0,1]) < 2C_5 L.$$

Applying Lemma 5.12 to X with $L' = L$, we have an $L_8 = L_8(L)$ so that

$$\ell_X(\gamma) < L_8.$$

It follows that

$$\ell_{X_h^\epsilon}(\gamma) < C_5 L_8.$$

Applying Lemma 5.12 to Z_X with $L' = C_5 L$, we have a $L_8' = L_8(C_5 L)$ for which

$$\ell_{Z_X}(\gamma) < L_8'$$

Letting

$$K_8 = \max\{L_8, L_8'\},$$

we may straighten f_θ^ϵ to send $\gamma \times \{0\}$ to its geodesic representative $\bar{\gamma} \subset X_h^\epsilon$ in X_h^ϵ and straighten further as before so that for $x \in \gamma$ the map f_θ^ϵ sends $x \times [0,1]$ to a geodesic. Arguing as for α_i, each annulus $\gamma_i \times [0,1]$ has mass

$$\operatorname{mass}_{f_\theta^\epsilon}(\gamma_i \times [0,1]) < 2K_8.$$

It follows that the union of the four annuli

$$\mathcal{T}_i = (A_i \times \{1\}) \cup (\alpha_i \times [0,1]) \cup (A_i \times \{0\}) \cup (\gamma_i \times [0,1])$$

along their boundaries is a torus with mass

$$\operatorname{mass}_{f_\theta^\epsilon}(\mathcal{T}_i) < 2K_8 + (8 + 2|\chi(S)|)\pi = K_8'.$$

Applying [Th2, Prop. 4.1], we have that the solid torus $\mathcal{V}_i \subset N$ bounded by $\mathcal{T}_i$ has mass

$$\mathrm{mass}_{f^\epsilon_\theta}(\mathcal{V}_i) \le \mathrm{mass}_{f^\epsilon_\theta}(\mathcal{T}_i) < K_8'.$$

It follows that the total f^ϵ_θ-mass of $\widehat{S} \times I$ is less than $3K_8'$, and since the number of pieces of the complement $S - P_X$ depends only on S, it follows that

$$\mathrm{mass}_{f^\epsilon_\theta}(\mathrm{cap}_X) < K_{\mathrm{cap}}$$

for an a priori constant K_{cap}.

∎

Conclusion. The proof of Proposition 5.8 is now an application of the preceding lemmas.

Proof: *(of Proposition 5.8).* Applying lemmas 5.9 and 5.11, we have

$$\begin{aligned} \mathrm{mass}_{f^\epsilon_\theta}(N) &= \mathrm{mass}_{f^\epsilon_\theta}(N_\Delta) + \mathrm{mass}_{f^\epsilon_\theta}(\mathrm{cap}_X) + \mathrm{mass}_{f^\epsilon_\theta}(\mathrm{cap}_Y) \\ &< K_\Delta \cdot \mathcal{V}_3 \cdot d_{\mathbf{P}}(P_X, P_Y) + 1 + 2K_{\mathrm{cap}}. \end{aligned}$$

Setting $K_\Delta \cdot \mathcal{V}_3 = K_3$ and $1 + 2K_{\mathrm{cap}} = K_4$, the result follows.

The case when $\partial S \neq \varnothing$. Minor modifications to the above arguments are required when $\partial S \neq \varnothing$, and $Q(X,Y)$ has peripheral rank-1 cusps. The primary difference in our model N in this case arises from the fact that boundary curves of S are not elements of the pants decompositions P_j interpolating between P_X and P_Y. For the purposes of the construction, triangulations suited to P_j are then triangulations of S with two vertices on each boundary component so that on each pair of pants in $S - P_j$ the triangulation has the structure of Definition 5.3. The blocks required to accomplish each elementary move are then identical; to map them in to $C(X,Y)$ we need only specify the behavior of the maps $f_{j,*}$ on the boundary curves.

Indeed, if $\gamma \subset \partial S$, there is no geodesic in the free homotopy class of γ. Choosing a horocycle h in the homotopy class of γ on the boundary $\partial(C(X,Y)_{\ge \epsilon_0})$ and a pair of antipodal vertices p_h and $\bar{p}_h$ on h, we let γ^* denote the piecewise geodesic obtained from straightening h *rel*-$p_h \cup \bar{p}_h$. The behavior of $f_{j,*}$ is then determined by requiring that $f_{j,*}$ send each $\gamma \subset \partial S$ to the corresponding γ^* and that $f_{j,*}$ be simplicial as before.

If $f_\Delta \colon N_\Delta \to C(X,Y)$ denotes the simplicial map of N_Δ into $C(X,Y)$ given by the above instructions, we modify f_Δ by sending ϵ_0 to 0: the edges and vertices of N_Δ incident on $\gamma \subset \partial S$ are mapped by f_Δ deeper and deeper into the γ-cusp so that the limiting map $f_{\Delta,0}$ sends each such tetrahedron to a tetrahedron or triangle with one vertex at infinity. The restriction of $f_{\Delta,0}$ to the interiors of $\partial^+ N_\Delta$ and $\partial^- N_\Delta$ factor through simplicial hyperbolic surfaces $h_X \colon Z_X \to C(X,Y)$ and $h_Y \colon Z_Y \to C(X,Y)$ realizing P_X and P_Y, and the restriction of $f_{\Delta,0}$ to N_Δ with the γ curves removed is a proper homotopy from h_X to h_Y.

The caps, now proper homotopies from Z_X to X^ϵ_h and Z_Y to Y^ϵ_h may then be added to the triangulated part. The total mass is again bounded by K_{cap}, since the caps still decompose into solid tori with boundary of bounded area (some are now bounded by properly embedded annuli asymptotic to cusps). The resulting map, a proper homotopy between the two components X^ϵ_h to Y^ϵ_h of $\partial \mathcal{N}_\epsilon(C(X,Y))$ may be spun about the spinning geodesics as before to force the Dehn twist tetrahedra to have arbitrarily small mass. Spinning sufficiently far, then, we again have the conclusion of Proposition 5.8.

■

6. Geometric limits

We conclude with the following application of our results to the study of algebraic and geometric limits of hyperbolic 3-manifolds.

Theorem 6.1. *Let $\{Q(X_k, Y_k)\}_{k=1}^{\infty} \subset AH(S)$ be an algebraically and geometrically convergent sequence with geometric limit N_G. Then N_G is geometrically finite if and only if there is a $K > 0$ for which*

$$d_{\mathrm{WP}}(X_k, Y_k) < K$$

for all k.

Proof: If the geometric limit N_G is geometrically infinite, the volumes of the convex cores of the approximates $Q(X_k, Y_k)$ grow without bound (see [CM, Lem. 7.1]), which, applying Theorem 1.2, implies that $d_{\mathrm{WP}}(X_k, Y_k)$ grows without bound.

Likewise, applying [Ta] or [Mc3, Thm. 3.1], when N_G is geometrically finite the ϵ-thick parts of the convex cores of $Q(X_k, Y_k)$ converge geometrically to that of N_G for ϵ sufficiently small. It follows that the volume of $\mathrm{core}(Q(X_k, Y_k))_{\geq \epsilon}$ converges to the volume of $\mathrm{core}(N_G)_{\geq \epsilon}$ for each sufficiently small ϵ and thus that

$$\mathrm{vol}(\mathrm{core}(Q(X_k, Y_k))) \to \mathrm{vol}(\mathrm{core}(N_G))$$

as k tends to ∞. The theorem follows.

■

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Abstract. We present a coarse interpretation of the Weil-Petersson distance $d_{\mathrm{WP}}(X, Y)$ between two finite area hyperbolic Riemann surfaces X and Y using a graph of pants decompositions introduced by Hatcher and Thurston. The combinatorics of the pants graph reveal a connection between Riemann surfaces and hyperbolic 3-manifolds conjectured by Thurston: the volume of the convex core of the quasi-Fuchsian manifold $Q(X, Y)$ with X and Y in its conformal boundary is comparable to the Weil-Petersson distance $d_{\mathrm{WP}}(X, Y)$. In applications, we relate the Weil-Petersson distance to the Hausdorff dimension of the limit set and the lowest eigenvalue of the Laplacian for $Q(X, Y)$, and give a new finiteness criterion for geometric limits.

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