Asymptotics of Weil-Petersson geodesics I: ending laminations, recurrence, and flows

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Abstract

We define an ending lamination for a Weil-Petersson geodesic ray. Despite the lack of a natural visual boundary for the Weil-Petersson metric [Br2], these ending laminations provide an effective boundary theory that encodes much of its asymptotic CAT(0) geometry. In particular, we prove an *ending lamination theorem* (Theorem 1.1) for the full-measure set of rays that recur to the thick part, and we show that the association of an ending lamination embeds asymptote classes of recurrent rays into the Gromov-boundary of the curve complex $\mathscr{C}(S)$. As an application, we establish fundamentals of the topological dynamics of the Weil-Petersson geodesic flow, showing density of closed orbits and topological transitivity.

Contents

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1 Introduction

This paper is the first in a series considering the asymptotics of geodesics in the Weil-Petersson metric on the Teichmüller space $\mathrm{Teich}(S)$ of a compact surface S with negative Euler characteristic.

In many settings, measured laminations and foliations encode the asymptotic geometry of Teichmüller space. As key examples, one has:

1. Thurston's natural compactification by projective measured laminations [Th3, Bon2],
2. invariant projective measured foliations for Teichmüller rays [Ker1], and
3. the parametrization of Bers's compactification by *end-invariants* (see [Min1, BCM]).

In a similar spirit, our goal will be to describe the asymptotics of Weil-Petersson geodesics in Teichmüller space by the use of laminations. We define a notion of an *ending lamination* for a Weil-Petersson geodesic ray (a geodesic from a point that leaves every compact subset of Teichmüller space) and investigate its role as an invariant for the ray. Since the Weil-Petersson metric is not complete, there are rays of finite Weil-Petersson length. These correspond to points in the Weil-Petersson completion, which is parametrized by products of lower dimensional Teichmüller spaces. Ending laminations for such rays are multi-curves with length functions tending to zero along the ray. Their initial tangents at a basepoint are dense in the unit tangent space [Br2], suggesting their associated multi-curves may play the role of "rational points" in encoding ending laminations for infinite rays.

We establish that the ending lamination is a complete asymptotic invariant for *recurrent* rays, namely, those rays whose projections to the moduli space $\mathscr{M}(S) = \mathrm{Teich}(S)/\mathrm{Mod}(S)$ (the quotient of Teichmüller space by the mapping class group) visit a fixed compact set at a divergent sequence of times. In particular, it follows that any two such rays starting at the same basepoint with the same ending lamination are identical up to parametrization. Despite the lack of naturality described in [Br2], this invariant allows us to establish fundamentals of the topological dynamics of the Weil-Petersson geodesic flow on the quotient $\mathscr{M}^1(S) = T^1\mathrm{Teich}(S)/\mathrm{Mod}(S)$ of the unit tangent bundle $T^1\mathrm{Teich}(S)$. We show

(I.) the set of closed Weil-Petersson geodesics is dense in $\mathscr{M}^1(S)$ (Theorem 1.6), and

(II.) there is a single Weil-Petersson geodesic that is dense in $\mathscr{M}^1(S)$ (Theorem 1.7).

To the extent the ending lamination determines the ray, one can employ properties of laminations to understand Weil-Petersson geometry. We prove

Theorem 1.1. (Recurrent Ending Lamination Theorem) *Let* $\mathbf{r}$ *be a recurrent Weil-Petersson geodesic ray in* $\text{Teich}(S)$ *with ending lamination* $\lambda(\mathbf{r})$. *If* $\mathbf{r}'$ *is any other geodesic ray with ending lamination* $\lambda(\mathbf{r}') = \lambda(\mathbf{r})$ *then* $\mathbf{r}'$ *is strongly asymptotic to* $\mathbf{r}$.

Here, we say $\mathbf{r}$ and $\mathbf{r}'$ are *strongly asymptotic* if there are parametrizations for which the distance between the rays satisfies

$$\lim_{t\to\infty} d(\mathbf{r}(t), \mathbf{r}'(t)) = 0.$$

In particular, the negative curvature of the Weil-Petersson metric guarantees that if $\mathbf{r}(0) = \mathbf{r}'(0)$, then the rays are identical if parametrized by arclength.

The *ending lamination* $\lambda(\mathbf{r})$ for a ray $\mathbf{r}$ arises out of the asymptotics of simple closed curves with an a priori length bound. Recall that by a theorem of Bers, there is a constant L_S depending only on S so that for each $X \in \text{Teich}(S)$ there is a pants decomposition by geodesics on X so that each such geodesic has length at most L_S. We call such a γ a *Bers curve* for X.

Given a Weil-Petersson geodesic ray $\mathbf{r}$, the *ending lamination* $\lambda(\mathbf{r})$ is a union of limits of Bers curves for surfaces $X_n = \mathbf{r}(t_n)$ along the ray. In section 2, we give a precise description and the proof that $\lambda(\mathbf{r})$ is well defined.

In Proposition 4.4, we show that for a recurrent ray $\mathbf{r}$, the ending lamination $\lambda(\mathbf{r})$ *fills* S. Thus, $\lambda(\mathbf{r})$ determines a point in $\mathscr{EL}(S)$, the Gromov boundary for the curve complex $\mathscr{C}(S)$ (see [MM1, Kla, Ham]). We remark that Theorem 1.1 determines a preferred subset $\mathscr{REL}(S) \subset \mathscr{EL}(S)$ corresponding to ending laminations for recurrent rays in the Weil-Petersson metric. In particular, the ending lamination determines whether or not a ray is recurrent.

Teichmüller geodesics. The *Teichmüller metric* is a $\text{Mod}(S)$-invariant Finsler metric on $\text{Teich}(S)$ measuring the minimal quasi-conformal distortion of the extremal quasi-conformal mapping between marked Riemann surfaces. We emphasize the distinctions of our settings from the more thoroughly studied behavior of Teichmüller geodesics.

In [Mas2], the second author shows that Teichmüller geodesics with the same vertical foliation are strongly asymptotic when the foliation is *uniquely ergodic* (meaning it admits a unique transverse invariant measure), and that if a Teichmüller geodesic ray is recurrent, then the vertical foliation is uniquely ergodic. By contrast, we note that there is no assumption of unique ergodicity for $\lambda(\mathbf{r})$ in Theorem 1.1 and that [Br3] presents examples of recurrent rays with non-uniquely ergodic laminations. Furthermore, these examples are sharp in the sense that without the assumption of recurrence examples are known of distinct infinite rays with the same filling ending lamination (see [Br3], and compare [Br2, §6]).

Visual boundaries. The negative curvature of the Weil-Petersson metric (see [Tro, Wol3]) provides for a compactification of $\mathrm{Teich}(S)$ by geodesic rays emanating from a fixed basepoint X, the *visual sphere* at X. Work of the first author (see [Br2]) demonstrates that the compactification of $\mathrm{Teich}(S)$ is basepoint dependent and, moreover, that the mapping class group fails to extend continuously to the compactification.

Standard arguments for topological transitivity and the density of closed orbits that arise in Riemannian manifolds of negative curvature involve the use of the boundary at infinity for the universal cover and the natural extension of the action of the fundamental group to the boundary.

The principal source of difficulty with carrying out such a line of argument here is precisely the source of the basepoint dependence shown in [Br2]. The lack of completeness of the metric gives rise to finite-length geodesic rays that leave every compact subset of Teichmüller space, and these *finite rays* determine a subset of the boundary on which the change of basepoint map is discontinuous. While such finite rays prevent the Weil-Petersson metric from exhibiting the more standard boundary structure arising in the setting of Hadamard manifolds (see [Eb]) we show the infinite length Weil-Petersson geodesic rays determine a natural *boundary at infinity* for the Weil-Petersson completion.

Theorem 1.2. (BOUNDARY AT INFINITY) *Let $X \in \mathrm{Teich}(S)$ be a basepoint.*

1. *For any $Y \in \mathrm{Teich}(S)$ with $Y \neq X$, and any infinite ray $\mathbf{r}$ based at X there is a unique infinite ray $\mathbf{r}'$ based at Y with $\mathbf{r}'(t) \in \mathrm{Teich}(S)$ for each t so that $\mathbf{r}'$ lies in the same asymptote class as $\mathbf{r}$.*

2. *The change of basepoint map restricts to a homeomorphism on the infinite rays.*

Though the Weil-Petersson completion $\overline{\mathrm{Teich}(S)}$ of $\mathrm{Teich}(S)$ does not satisfy the extendability of geodesics requirement for a standard notion of a $\mathrm{CAT}(0)$ boundary to be well defined, one can simply restrict attention to the infinite rays and consider asymptote classes of infinite rays in the completion of the Weil-Petersson metric, where two infinite rays are in the same *asymptote class* if they lie within some bounded Hausdorff distance of one another. Theorem 1.2 gives a basepoint-independent topology on these asymptote classes, and we denote the resulting space by $\partial_\infty\overline{\mathrm{Teich}(S)}$.

Any flat subspace in a $\mathrm{CAT}(0)$ space provides an obstruction to the visibility property exhibited in strict negative curvature, namely, the existence of a single bi-infinite geodesic asymptotic to any two distinct points at infinity. The encoding guaranteed by Theorem 1.1 of recurrent rays via laminations remedies this conclusion to some degree, as it guarantees such a visibility property almost everywhere with respect to Riemannian volume measure on the unit tangent bundle.

Theorem 1.3. (RECURRENT VISIBILITY) *Let* $\mathbf{r}^+$ *and* $\mathbf{r}^-$ *be two distinct infinite rays based at* X.

1. *If* $\mathbf{r}^+$ *is recurrent, then there is a single bi-infinite geodesic* $\mathbf{g}(t)$ *so that* $\mathbf{g}^+ = \mathbf{g}|_{[0,\infty)}$ *is* strongly *asymptotic to* $\mathbf{r}^+$ *and* $\mathbf{g}^- = \mathbf{g}|_{(-\infty,0]}$ *is asymptotic to* $\mathbf{r}^-$. *In particular, if both* $\mathbf{r}^+$ *and* $\mathbf{r}^-$ *are recurrent, then* $\mathbf{g}$ *is strongly asymptotic to both* $\mathbf{r}^-$ *and* $\mathbf{r}^+$.

2. *If* μ *in the measured lamination space* $\mathscr{ML}(S)$ *has bounded length on* $\mathbf{r}^\pm$ *then it has bounded length on* $\mathbf{g}^\pm$.

Theorem 1.2 leads one naturally to the question of whether, as in other compactifications of Teichmüller space, the laminations associated to rays serve as parameters. Applying Theorem 1.1, we find that such a parametrization holds for the recurrent locus.

Corollary 1.4. *The map* λ *that associates to an equivalence class of recurrent rays its ending lamination is a homeomorphism to the subset* $\mathscr{REL}(S)$ *in* $\mathscr{EL}(S)$.

We note that examples of [Br3] show this parametrization fails in general, even when the ending lamination is filling.

To describe our strategy further, we review geometric aspects of the Weil-Petersson metric and its completion.

Weil-Petersson geometry. The Weil-Petersson metric g_{WP} on Teich(S) arises from the L^2 inner product

$$\langle \varphi, \psi \rangle_{\mathrm{WP}} = \int_X \frac{\varphi\overline{\psi}}{\rho^2}$$

on the cotangent space $Q(X) = T^*_X\mathrm{Teich}(S)$ to Teichmüller space, naturally the holomorphic quadratic differentials on X, where $\rho(z)|dz|$ is the hyperbolic metric on X.

A fundamental distinction between the Weil-Petersson metric and other metrics on Teichmüller space is its lack of completeness, due to Wolpert and Chu [Wol1, Chu]. It is nevertheless geodesically convex [Wol4], and has negative sectional curvatures [Tro, Wol3].

The failure of completeness corresponds precisely to *pinching paths* in Teich(S) along which a simple closed geodesic on X is pinched to a cusp. It is due to the second author that the completion $\overline{\mathrm{Teich}(S)}$ is identified with the *augmented Teichmüller space* and is obtained by adjoining noded Riemann surfaces as limits of such pinching paths [Mas1]. Via this identification, then, the completion $\overline{\mathrm{Teich}(S)}$ (with its extended metric) descends to a metric on the Mumford-Deligne compactification $\overline{\mathscr{M}(S)}$ of the moduli space (cf. [Ab, Brs]).

The *Weil-Petersson geodesic flow* on the unit tangent bundle $T^1\mathrm{Teich}(S)$ is the usual geodesic flow in the sense of Riemannian manifolds with respect to the Weil-Petersson metric. It commutes with the isometric action of the modular group $\mathrm{Mod}(S)$ and defines a flow on $\mathscr{M}^1(S)$.

Because of failure of completeness, however, the geodesic flow is not everywhere defined on $\mathscr{M}^1(S)$; some directions meet the compactification within finite Weil-Petersson distance. The situation is remedied by the following.

Proposition 1.5. *The geodesic flow is defined for all time on a full measure subset of $\mathscr{M}^1(S)$, with respect to Liouville measure.*

As a consequence, we address the question of the topological dynamics of the geodesic flow on $\mathscr{M}^1(S)$.

The fact that the recurrent rays have full measure in the visual sphere allows us to approximate directions in the unit tangent bundle arbitrarily well by recurrent directions. As a consequence, we have

Theorem 1.6. (CLOSED ORBITS DENSE) *The set of closed Weil-Petersson geodesics is dense in $\mathscr{M}^1(S)$.*

Applying our parametrization by ending laminations of the boundary at infinity, we may use the stable and unstable laminations for the axes of pseudo-Anosov isometries of $\mathrm{Teich}(S)$ to find based at any X a geodesic ray whose projection to $\mathscr{M}(S)$ has a dense trajectory in $\mathscr{M}^1(S)$.

Theorem 1.7. (DENSE GEODESIC) *There is a dense Weil-Petersson geodesic in $\mathscr{M}^1(S)$.*

Combinatorics of Weil-Petersson geodesics. While the this paper's focus on recurrence establishes the importance of the ending lamination as a tool to analyze Weil-Petersson geodesics, it does not directly address the connection between the combinatorics of the lamination (in the sense of [MM2]) and the geometry of geodesics.

We take up this discussion in [BMM] to prove a *bounded geometry theorem* relating bounded geometry (a lower bound for the injectivity radius of surfaces along the geodesic) to a bounded combinatorics condition analogous to bounded continued fractions, and vice versa. These results give good control over the subset of geodesics with bounded geometry, and imply further dynamical consequences involving the topological entropy of the geodesic flow on compact invariant subsets. The analogous discussion for the Teichmüller flow has been carried out by K. Rafi [Raf], who obtains a complete description of the list of short curves along a Teichmüller geodesic in terms of the vertical and horizontal foliations.

We expect in general that the ending lamination should predict extensive information about bounded and short curves along the ray, in line with the ending lamination theorem of [BCM]. In particular, we make the following conjecture.

Conjecture 1.8. (SHORT CURVES) *Let* $\mathbf{g}$ *be a bi-infinite Weil-Petersson geodesic with ending laminations* λ^- *and* λ^+ *that fill the surface S, and let* $M \cong S \times \mathbb{R}$ *be a totally degenerate hyperbolic 3-manifold with ending laminations* λ^- *and* λ^+*. Then we have*

1. *for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *so that for each simple closed curve* γ *on S, if* $\inf_t \ell_\gamma(\mathbf{g}(t)) < \delta$ *then* $\ell_\gamma(M) < \varepsilon$.
2. *for each* $\delta' > 0$ *there is an* $\varepsilon' > 0$ *so that for each simple closed curve* γ *on S, if* $\ell_\gamma(M) < \varepsilon'$ *then* $\inf_t \ell_\gamma(\mathbf{g}(t)) < \delta'$.

Here, $\ell_\gamma(M)$ denotes the arclength of the unique geodesic representative of γ in M. Though the present paper will not treat them in more detail, hyperbolic 3-manifolds and Kleinian groups are discussed in in [Th1, Bon1, Mc, Min1] among other places. The conjecture is essentially a combinatorial one, as the geometry of hyperbolic 3-manifolds was shown to be controlled by the combinatorics of the curve complex in [Min2, BCM].

Such expected connections with ends of hyperbolic 3-manifolds motivate other questions about the structure of the ending lamination $\lambda(\mathbf{r})$ for a ray $\mathbf{r}$.

Conjecture 1.9. *Let* $\mathbf{r}$ *be a Weil-Petersson geodesic ray along which no simple closed curve has length asymptotic to zero. Then the ending lamination* $\lambda(\mathbf{r})$ *fills the surface.*

We establish this conjecture for recurrent rays in Proposition 4.4.

Plan of the paper. In section 2 we set out necessary background, and give the definition of ending lamination for a Weil-Petersson geodesic ray, establishing its basic properties. Section 3 establishes that the geodesic flow is defined for all time on a full measure set and gives the natural application of the Poincaré recurrence theorem in this setting. Section 4 establishes the main theorem, that the ending lamination is a complete invariant for a recurrent ray, as well deriving important topological properties of the ending lamination itself that mirror the behavior of ending laminations for hyperbolic 3-manifolds. Finally, in section 5 we present applications of this boundary theory to the topological dynamics of the Weil-Petersson geodesic flow.

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2 Ending laminations for Weil-Petersson rays

In this section we begin by reviewing some of the notions and results necessary for our discussion, provide references for background, and give the definition of the ending lamination, establishing its basic properties.

Teichmüller space and moduli space. The Teichmüller space of S, $\text{Teich}(S)$, parametrizes the marked, complete, finite-area hyperbolic structures on $\text{int}(S)$, namely, pairs (f, X) where

$$f\colon \text{int}(S) \to X$$

is a *marking homeomorphism* to a finite-area hyperbolic surface X and $(f, X) \sim (g, Y)$ if there is an isometry $\phi\colon X \to Y$ for which $\phi \circ f$ is isotopic to g. The *mapping class group* $\text{Mod}(S)$ of orientation preserving homeomorphisms up to isotopy acts naturally on $\text{Teich}(S)$ by precomposition of markings, inducing an action by isometries in the Weil-Petersson metric. The quotient is the *moduli space* $\mathscr{M}(S)$, of hyperbolic structures on $\text{int}(S)$ (without marking), and the Weil-Petersson metric descends to a metric on $\mathscr{M}(S)$.

Hyperbolic geometry of surfaces. Let $\mathcal{S}$ denote the collection of isotopy classes of essential, non-peripheral simple closed curves on S. A *pants decomposition* P is a maximal collection of distinct elements of $\mathcal{S}$ with $i(\alpha, \beta) = 0$ for any α and β in P. Here, $i\colon \mathcal{S} \times \mathcal{S} \to \mathbb{Z}$ denotes the *geometric intersection number* which counts the minimal number of intersections between representatives of the isotopy classes α and β on S. Given $X \in \text{Teich}(S)$, each $\alpha \in \mathcal{S}$ has a unique geodesic representative α^* on X. Its arclength determines a *geodesic length function*

$$\ell_\alpha\colon \text{Teich}(S) \to \mathbb{R}_+.$$

In [Wol4], Wolpert proved that along a geodesic $\mathbf{g}(t)$ the length function $\ell_\alpha(\mathbf{g}(t))$ is strictly convex.

For all that follows it will be important to have in place the Theorem of Bers (see [Bus]) that given S, a compact orientable surface of negative Euler characteristic, there is a constant $L_S > 0$ so that for each $X \in \text{Teich}(S)$ there is a pants decomposition P_X determined by simple closed geodesics on X so that

$$\ell_\gamma(X) < L_S$$

for each $\gamma \in P_X$. We call the pants decomposition P_X a *Bers pants decomposition for* X and the curves in such a pants decomposition P_X *Bers curves for* X.

A *geodesic lamination* λ on a hyperbolic surface $X \in \text{Teich}(S)$ is a closed subset of X foliated by simple complete geodesics. Employing the natural boundary at infinity for $\widetilde{X}$, a geodesic lamination, like a simple closed curve, has a well defined isotopy class on X, and we may speak of a single geodesic lamination λ as an object associated to S with realizations on each hyperbolic structure $X \in \text{Teich}(S)$ (see [Th1, Ch. 8], [Ha], and [Bon2]). The realizations of geodesic laminations on X may be given the Hausdorff topology, and the correspondence between realizations of λ on different surfaces X and X' gives a homeomorphism. Hence, we refer to a single *geodesic lamination space* $\mathscr{GL}(S)$.

A geodesic lamination λ equipped with a *transverse measure* μ, namely a measure on each arc transverse to the leaves of λ invariant under isotopy preserving intersections with λ, determines a *measured lamination*. The lamination λ is called the *support* of the measured lamination μ and is denoted by $|\mu|$. The simple closed curves with positive real weights play the role of Dirac measures, and the measured lamination space $\mathscr{ML}(S)$ is identified with the closure of the image of the embedding $\iota\colon \mathcal{S} \times \mathbb{R}_+ \to \mathbb{R}_+^{\mathcal{S}}$ by

$$\langle \iota(\alpha, t)\rangle_\beta = t \cdot i(\alpha, \beta).$$

(see [FLP, Th1, Bon2]). The natural action of $\mathbb{R}^+$ on $\mathscr{ML}(S)$ by scalar multiplication of transverse measures gives rise to Thurston's projective measured lamination space $\mathscr{PML}(S) = (\mathscr{ML}(S) - \{0\})/\mathbb{R}_+$. Throughout, $[\mu]$ will denote the projective class in $\mathscr{PML}(S)$ of a nonzero measured lamination $\mu \in \mathscr{ML}(S)$.

The geodesic length function for a simple closed curve extends to a bi-continuous function

$$\ell_{.}(.)\colon \mathscr{ML}(S) \times \text{Teich}(S) \to \mathbb{R}_+$$

by defining $\ell_{t\cdot\gamma}(X) = t(\ell_\gamma(X))$ for $t \in \mathbb{R}_+$ and $\gamma \in \mathcal{S}$, and setting

$$\ell_\mu(X) = \lim_{n\to\infty} s_n(\ell_{\gamma_n}(X))$$

(see [Ker2, Bon2]). Wolpert strengthens his convexity result for simple closed curves to apply to this "total length" of a measured lamination (see [Wol6])

Theorem 2.1 (Wolpert)**.** *Given a Weil-Petersson gedesic $\mathbf{g}(t)$, the length $\ell_\mu(\mathbf{g}(t))$ of a measured lamination is a strictly convex function of t.*

Curve and arc complexes. The complex of curves $\mathscr{C}(S)$ associated to the surface S is a simplicial complex whose vertices are elements of $\mathcal{S}$, and whose k-simplices span $k+1$-tuples of vertices whose corresponding isotopy classes can be realized as a pairwise disjoint collection of simple closed curves on S. By convention, we obtain the *augmented curve complex* by adjoining the empty simplex and denote

$$\widehat{\mathscr{C}(S)} = \mathscr{C}(S) \cup \varnothing.$$

It was shown in [MM1] that the curve complex $\mathscr{C}(S)$ is a δ-hyperbolic path metric space. Any such space carries a natural *Gromov boundary*, which is identified with asymptote classes of quasigeodesic rays where two rays are *asymptotic* if they lie within uniformly bounded Hausdorff distance. Klarreich showed [Kla] (see also [Ham]) that the Gromov boundary is identified with the space $\mathscr{E}\mathscr{L}(S)$ of geodesic laminations that arise as supports of *filling* measured laminations. (A lamination $\mu \in \mathscr{M}\mathscr{L}(S)$ is filling if every simple closed curve γ satisfies $i(\mu, \gamma) > 0$). The space $\mathscr{E}\mathscr{L}(S)$ inherits the quotient topology from $\mathscr{M}\mathscr{L}(S)$, but it is a Hausdorff subspace of this quotient; this topology is sometimes called the *measure-forgetting topology* or the *Thurston topology* [CEG].

Given a reference hyperbolic structure $X \in \operatorname{Teich}(S)$, for each $\gamma \in \mathcal{S}$ there is a $\delta_\gamma > 0$ such that the neighborhood $\mathscr{N}_{\delta_\gamma}(\gamma^*)$ of the geodesic representative γ^* on X is a regular neighborhood, and so that for each η with $i(\eta, \gamma) = 0$, we have disjoint neighborhoods

$$\mathscr{N}_{\delta_\gamma}(\gamma^*) \cap \mathscr{N}_{\delta_\eta}(\eta^*) = \emptyset.$$

Given a simplex $\sigma \subset \mathscr{C}(S)$, denote by σ^0 the set of vertices of σ and let $\mathbf{collar}(\sigma)$ be the union

$$\bigcup_{\gamma \in \sigma^0} \mathscr{N}_{\delta_\gamma}(\gamma^*).$$

Fixing this notation, we make the following definition.

Definition 2.2. *Let λ be a connected geodesic lamination. The* supporting subsurface *$S(\lambda) \subset S$ is the compact subsurface up to isotopy represented by the smallest subsurface $Y \subset X$ containing the realization of λ as a geodesic lamination on X, whose non-peripheral boundary curves are a union of curves in* $\mathbf{collar}(\sigma)$ *for some $\sigma \in \widehat{\mathscr{C}(S)}$.*

The pants complex. A quasi-isometric model was obtained for the Weil-Petersson metric in [Br1] using pants decompositions of surfaces. We say two pants decompositions P and P' are related by an *elementary move* if P' is obtained from P by replacing a curve α in P with a curve β in such a way that $i(\alpha, \beta)$ is minimized. Let $P(S)$ denote the graph whose vertices represent distinct isotopy classes of pants decompositions of S, or maximal simplices in $\mathscr{C}(S)$, and whose edges join vertices that differ by an elementary move.

Hatcher and Thurston showed that $P(S)$ is connected (see [HLS]) so we may consider the edge metric on $P(S)$ as a distance on the pants decompositions of S. Letting $Q \colon P(S) \to \operatorname{Teich}(S)$ be any map that associates to P a surface X on which P is a Bers pants decomposition.

Theorem 2.3. ([Br1, Thm. 1.1]) *The map Q is a quasi-isometry.*

In other words, the map Q distorts distances by a bounded multiplicative factor and a bounded additive constant.

The Weil-Petersson completion and its strata. The non-completeness of the Weil-Petersson metric corresponds to finite-length paths in Teichmüller space along which length functions for simple closed curves converge to zero. In [Mas1], the completion is described concretely as the *augmented Teichmüller space* [Brs, Ab] obtained from Teichmüller space by adding strata consisting of spaces $\mathscr{S}_\sigma$ defined by the vanishing of length functions

$$\ell_\alpha \equiv 0$$

for each $\alpha \in \sigma^0$ where σ is a simplex in the augmented curve complex $\widehat{\mathscr{C}(S)}$. Points in the σ-null strata $\mathscr{S}_\sigma$ correspond to *nodal* Riemann surfaces Z, where (paired) cusps are introduced along the curves in σ^0.

One can describe the topology via extended Fenchel Nielsen coordinates: Given a pants decomposition P, the usual coordinates map $\mathrm{Teich}(S)$ to $\prod_{\gamma\in P}\mathbb{R}\times\mathbb{R}^+$, where the first coordinate of each pair measures twist and the second is the length function of the corresponding vertex of P. We extend this to allow length 0, and take the quotient by identifying $(t,0)\sim(t',0)$ in each $\mathbb{R}\times\mathbb{R}^+$ factor. The topology near any point of a stratum $\mathscr{S}_\sigma$, where $\sigma^0\subset P$, is such that this map is a homeomorphism near that point.

Then the strata $\mathscr{S}_\sigma$ are naturally products of lower dimensional Teichmüller spaces corresponding to the complete, finite-area hyperbolic "pieces" of the nodal surface $Z\in\mathscr{S}_\sigma$.

As observed in [Wol5, MW] The completion $\overline{\mathrm{Teich}(S)}$ has the structure of a CAT(0) space: it is a length space, satisfying the sub-comparison property for chordal distances in comparison triangles in the Euclidean plane (see [BH, II.1, Defn. 1.1]). Given $(X,Y)\in\overline{\mathrm{Teich}(S)}\times\overline{\mathrm{Teich}(S)}$ we will denote by $\overline{XY}$ the unique Weil-Petersson geodesic joining X to Y. Then the *main stratum*, $\mathscr{S}_\varnothing$, is simply the full Teichmüller space $\mathrm{Teich}(S)$.

Apropos of this convention, we recall the fundamental non-refraction for geodesics on the Weil-Petersson completion.

Theorem 2.4 ([DW, Wol5]). (NON-REFRACTION IN THE COMPLETION) *Let $\overline{XY}$ be the geodesic joining X and Y in $\overline{\mathrm{Teich}(S)}$, and let σ_- and σ_+ be the maximal simplices in the curve complex so that $X\in\mathscr{S}_{\sigma_-}$ and $Y\in\mathscr{S}_{\sigma_+}$. If $\eta=\sigma_-\cap\sigma_+$, then we have*

$$\mathrm{int}(g)\subset\mathscr{S}_\eta.$$

We remark that in the special case that X and Y lie in the interior of Teichmüller space the theorem is simply a restatement of Wolpert's geodesic convexity theorem (see [Wol4]).

A consequence of Theorem 2.4 is a classification of elements of $\text{Mod}(S)$ in terms of their action by isometries of the Weil-Petersson completion $\overline{\text{Teich}(S)}$ (see [DW, Wol5]). In particular, a mapping class ψ is *pseudo-Anosov* if no non-zero power of ψ preserves any isotopy class of simple closed curves on S. As in the setting of the Teichmüller metric, ψ preserves an invariant Weil-Petersson *geodesic axis* $A_\psi \subset \text{Teich}(S)$ on which it acts by translation.

Weil-Petersson geodesic rays and ending laminations. Allowing $\omega = \infty$, a Weil-Petersson geodesic ray is a geodesic

$$\mathbf{r}\colon [0,\omega) \to \text{Teich}(S)$$

parametrized by arclength, so that $\mathbf{r}(t)$ leaves every compact subset of Teichmüller space. Note this means that even when $\omega < \infty$, the ray cannot be extended further.

Although triangles in a $\text{CAT}(0)$ space can fail the stronger thin-triangles condition of Gromov hyperbolicity, the comparison property for triangles suffices to guarantee that there is still a well defined notion of an *asymptote class* for a geodesic ray: two rays $\mathbf{r}$ and $\mathbf{r}'$ lie in the same asymptote class, or are *asymptotic* if there is a $D > 0$ so that

$$d(\mathbf{r}(t), \mathbf{r}'(t)) < D$$

for each t.

Fixing a basepoint $X \in \text{Teich}(S)$, however, it is natural in the setting of negative curvature to consider the sphere of geodesic rays emanating from X ($\mathbf{r}(0) = X$), which we denote by $\mathscr{V}_X(S)$, or the Weil-Petersson *visual sphere*. Geodesic convexity (see [Wol4]) guarantees that we can compactify Teichmüller space by appending $\mathscr{V}_X(S)$.

We call a simple closed curve $\gamma \in \mathcal{S}$ a *Bers curve for the ray* $\mathbf{r}$ if there is a $t \in [0,\omega)$ for which γ is a Bers curve for $\mathbf{r}(t)$.

We associate a geodesic lamination $\lambda(\mathbf{r})$ to a ray $\mathbf{r}$ as follows.

Definition 2.5. *An* ending measure *for a geodesic ray* $\mathbf{r}(t)$ *is any representative* $\mu \in \mathscr{ML}(S)$ *of a limit of projective classes* $[\gamma_n] \in \mathscr{PML}(S)$ *of any infinite sequence of distinct Bers curves for* $\mathbf{r}$.

Remark. The definition of ending measures parallels Thurston's definition of the ending lamination for a simply degenerate end of a hyperbolic 3-manifold (see [Th1, Ch. 9]). We remark that is possible for an ending measures to be supported on a subsurface of $R \subset S$, while the geometry of the complement of R stabilizes along $\mathbf{r}$. This explains the use of infinite sequences of Bers curves rather than Bers pants decompositions in the definition, since these may intersect non-trivially in a subsurface whose geometry is converging.

Given $L > 0$ there may be a fixed curve γ that satisfies $\ell_\gamma(\mathbf{r}(t)) \le L$ for each t. Those γ that have no positive lower bound to their length, however, play a special role.

Definition 2.6. *A simple closed curve γ is a* pinching curve *for* $\mathbf{r}$ *if* $\ell_\gamma(\mathbf{r}(t)) \to 0$ *as* $t \to \omega$*.*

A single ray can exhibit both types of behavior, motivating the following definition.

Definition 2.7. *If* $\mathbf{r}(t)$ *is a Weil-Petersson geodesic ray, the* ending lamination $\lambda(\mathbf{r})$ *for* $\mathbf{r}$ *is the union of the pinching curves and the geodesic laminations arising as supports of ending measures for* $\mathbf{r}$*.*

To justify the definition we must show that pinching curves and supports of ending measures together have the underlying structure of a geodesic lamination. Specifically, we must show that pinching curves and ending measures have no transverse intersections, or that $i(\mu_1, \mu_2) = 0$ for any pair of pinching curves or ending measures.

We first establish the following basic property of ending measures.

Lemma 2.8. *If a* $\mathbf{r}$ *has finite length then its collection of ending measures is empty.*

Proof. It suffices to show that if $\mathbf{r}$ has finite length then there does not exist an infinite sequence of distinct Bers curves.

But a finite-length ray $\mathbf{r}(t)$ converges to a nodal surface Z in the Weil-Petersson completion $\overline{\mathrm{Teich}(S)}$, and for each simple closed curve γ on S either

1. there is a pinching curve α for which $i(\alpha, \gamma) > 0$, or
2. the length of γ converges along the ray $\mathbf{r}(t)$ to its length on Z.

In the first case, the length of γ on $\mathbf{r}(t)$ diverges as $t \to \omega$ by the collar lemma (see [Bus]). It follows that the union of Bers curves over all surfaces $\mathbf{r}(t)$ is finite. □

Theorem 2.1 guarantees that each pinching curve γ for $\mathbf{r}$ has length decreasing in t. By showing their boundedness along infinite rays, we may apply Theorem 2.1 again to see the same holds for ending measures.

Lemma 2.9. *Let* μ *be any ending measure for* $\mathbf{r}$*. Then* $\ell_\mu(\mathbf{r}(t))$ *is decreasing in* t*.*

Proof. Assume $\mathbf{r}$ is based at $X \in \mathrm{Teich}(S)$. Let γ_n be a sequence of Bers curves for the ray $\mathbf{r}$ so that the length of γ_n is infimized at $\mathbf{r}(t_n)$, and for which $t_i < t_{i+1}$, $i \in \mathbb{N}$. Let $[\mu]$ be any accumulation point of the sequence of projective classes $[\gamma_n]$ in $\mathscr{PML}(S)$. Then μ is an ending measure for $\mathbf{r}$. We may assume, after rescaling, that μ is the representative in the projective class $[\mu]$ with $\ell_\mu(X) = 1$.

Letting $s_n > 0$ be taken so that

$$s_n = \frac{1}{\ell_{\gamma_n}(X)},$$

the measured laminations $s_n\gamma_n$ satisfy $\ell_{s_n\gamma_n}(X) = 1$ for each n, and it follows that $s_n\gamma_n \to \mu$ in $\mathscr{ML}(S)$.

Fixing a value $t' > 0$, there is an N' so that for $n > N'$, we have $t_n > t'$. Applying strict convexity of the length function $\ell_{\gamma_n}(\mathbf{r}(t))$ as a function of t, [Wol4], we conclude that

$$\ell_{s_n\gamma_n}(\mathbf{r}(t')) < 1$$

for each $n > N'$. We conclude that

$$\ell_\mu(\mathbf{r}(t')) \leq 1.$$

Since $t' > 0$ was arbitrary, and $\ell_\mu(\mathbf{r}(t))$ is a strictly convex function of t by Theorem 2.1, we conclude that $\ell_\mu(\mathbf{r}(t))$ is decreasing in t. $\square$

For future reference, we establish the following continuity property for the behavior of bounded length laminations along rays.

Lemma 2.10. *Let $\mathbf{r}_n \to \mathbf{r}$ be a convergent sequence of rays in the visual sphere $\mathscr{V}_X(S)$. Then if μ_n is any sequence of ending measures or weighted pinching curves for $\mathbf{r}_n$, any representative $\mu \in \mathscr{ML}(S)$ of the limit $[\mu]$ of projective classes $[\mu_n]$ in $\mathscr{PML}(S)$ has bounded length along the ray* $\mathbf{r}$.

Proof. After normalizing so that $\ell_{\mu_n}(X) = 1$ we may assume that $\ell_{\mu_n}(\mathbf{r}_n(t)) \leq 1$ along $\mathbf{r}_n$. Then for each surface $Y = \mathbf{r}(s)$ along $\mathbf{r}$ there are surfaces $X_n = \mathbf{r}_n(s)$ with $X_n \to Y$ in $\mathrm{Teich}(S)$. Then $\ell_{\mu_n}(X_n) \to \ell_\mu(Y)$ and thus we have $\ell_\mu(Y) \leq 1$. Since s is arbitrary, the Lemma follows. $\square$

Proposition 2.11. *Given a ray $\mathbf{r}$, the union $\lambda(\mathbf{r})$ is a non-empty geodesic lamination.*

Proof. We first show that given $\mathbf{r}$, there exists either a pinching curve or an ending measure for $\mathbf{r}$. If $\mathbf{r}$ is a ray of finite length, then it terminates in the completion at a nodal surface Z in a boundary stratum $\mathscr{S}_\sigma$. It follows that each curve γ associated to a vertex of σ has length tending to zero along $\mathbf{r}$ and is thus a pinching curve for $\mathbf{r}$.

Assume there are no pinching curves for $\mathbf{r}$. Then, since $\mathbf{r}$ leaves every compact subset of $\mathrm{Teich}(S)$, and it does not terminate in the completion, it follows that it has infinite Weil-Petersson length. Then we claim there is a non-zero ending measure μ for $\mathbf{r}$. It suffices to show that there are infinitely many distinct Bers curves γ_n for surfaces $\mathbf{r}(t_n)$, with $t_n \to \infty$. But otherwise, the set of all Bers pants decompositions along the ray is also finite. By Theorem 2.3, we obtain a bound for the length of the ray $\mathbf{r}$ via the quasi-isometry Q, contradicting the assumption that $\mathbf{r}$ was infinite.

As in the definition of the ending lamination for hyperbolic 3-manifolds [Th1, Ch. 8], it suffices to show that for any pair μ_1 and μ_2 of weighted pinching curves or ending measures, that the intersection number satisfies

$$i(\mu_1, \mu_2) = 0.$$

We note first that by the collar lemma any two pinching curves for $\mathbf{r}$ must be disjoint. Furthermore, if γ is a pinching curve for $\mathbf{r}$, then γ is disjoint from each Bers curve on $\mathbf{r}(t)$ for t sufficiently large. Thus, if μ is an ending measure for $\mathbf{r}(t)$, then we have $i(\gamma, \mu) = 0$ as well. Thus we reduce to the case that μ_1 and μ_2 are both ending measures.

Assume that $i(\mu_1, \mu_2) > 0$. We note in particular that if μ_1 and μ_2 fill the surface, Lemma 2.9 guarantees that the ray $\mathbf{r}(t)$ defines a path of surfaces that range in a compact family in Teich(S) by Thurston's *Binding Confinement* (see [Th2, Prop. 2.4]). This contradicts the assumption that $\mathbf{r}$ leaves every compact subset of Teich(S).

More generally, let μ_1 and μ_2 fill a proper essential subsurface $Y \subset S$. Then a more general version of binding confinement, *Converge on Subsurface* (see [Th2, Thm. 6.2]), together with Lemma 2.9 ensures that the representations $\rho_t \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ for which $\mathbf{r}(t) = \mathbb{H}^2/\rho_t(\pi_1(S))$ have restrictions to $\pi_1(Y)$ that converge up to conjugacy after passing to a subsequence.

It follows, for any curve $\eta \in \mathscr{C}(Y)$, that the length $\ell_\eta(\mathbf{r}(t))$ is bounded away from zero and infinity. In particular Y contains no pinching curves. By the collar lemma η has a collar neighborhood of definite width in each $\mathbf{r}(t)$, which implies for any sequence γ_n of Bers curves on $\mathbf{r}(t_n)$, that $i(\eta, \gamma_n)$ is bounded above.

Each ending measure μ_i is a limit of weighted Bers curves $s_n\gamma_n$, with $s_n \to 0$, so it follows that for each $\eta \in \mathscr{C}(Y)$ we have

$$i(\eta, \mu_i) = \lim_{n\to\infty} i(\eta, s_n\gamma_n) = 0,$$

which contradicts that the support of μ_i intersects Y.

We conclude that $i(\mu_1, \mu_2) = 0$, and thus that the set of complete geodesics in the support of all pinching curves and ending measures forms a closed subset consisting of disjoint complete geodesics, namely, a geodesic lamination. $\square$

We note the following corollary of the proof.

Corollary 2.12. *Let $\mu \in \mathscr{ML}(S)$ be any lamination whose length is bounded along the ray $\mathbf{r}$. Then if μ' is an ending measure for $\mathbf{r}$ or a measure on any pinching curve for $\mathbf{r}$, we have*

$$i(\mu, \mu') = 0.$$

Proof. The proof of Proposition 2.11 employs only the bound on the length of μ_1 and μ_2 along the ray to show the vanishing of their intersection number. The argument applies equally well under the assumption that μ_1 is a simple closed curve of bounded length, and μ_2 is an ending measure, or a weighted pinching curve. Letting μ play the role of μ_1 and μ' play the role of μ_2, the Corollary follows. $\square$

By Thurston's classification of elements of $\mathrm{Mod}(S)$, a pseudo-Anosov element $\psi \in \mathrm{Mod}(S)$ determines laminations μ^+ and μ^- in $\mathscr{ML}(S)$, invariant by ψ up to scale [Th3]. Each determines an unique projective class in $\mathscr{PML}(S)$, the so-called *stable* and *unstable* laminations for ψ, and arises as a limit of iteration of ψ on $\mathscr{PML}(S)$. Specifically, given a simple closed curve γ, we have

$$[\mu^+] = \lim \psi^n([\gamma]) \quad \text{and} \quad [\mu^-] = \lim \psi^{-n}([\gamma])$$

in $\mathscr{PML}(S)$. Similarly, each $X \in A_\psi$, the axis of ψ, determines a *forward ray* $\mathbf{r}^+$ based at X so that $\psi(\mathbf{r}^+) \subset \mathbf{r}^+$ and a *backward ray* $\mathbf{r}^-$ at X so that $\mathbf{r}^- \subset \psi(\mathbf{r}^-)$. Invariance of the axis A_ψ, then, immediately gives the following relationship between the stable and unstable laminations for ψ and the ending laminations for the forward and backward rays at X for the invariant axis A_ψ.

Proposition 2.13. *Let $\psi \in \mathrm{Mod}(S)$ be a pseudo-Anosov element with invariant axis A_ψ. Let $X \in A_\psi$, and let $\mathbf{r}^+$ and $\mathbf{r}^-$ be the forward and backward geodesic rays at X determined by A_ψ. Then we have*

$$|\mu^+| = \lambda(\mathbf{r}^+) \quad \textit{and} \quad |\mu^-| = \lambda(\mathbf{r}^-)$$

where μ^+ is the stable lamination for ψ and μ^- is the unstable lamination.

Proof. Letting γ be a Bers curve for the surface X, the projective class $[\mu^+]$ of μ^+ is the limit of the projective classes $[\gamma_n]$ where $\gamma_n = \psi^n(\gamma)$ and likewise, $[\mu^-]$ is the limit of $[\gamma_{-n}]$. Since γ_n is a Bers curve for $\psi^n(X)$, it follows that μ^+ and μ^- are ending measures $\mathbf{r}^+$ and $\mathbf{r}^-$, respectively. Since μ^+ fills the surface, any other ending measure μ for $\mathbf{r}^+$ has intersection number $i(\mu^+, \mu) = 0$, so we have $\lambda(\mathbf{r}^+) = |\mu^+|$ and likewise $\lambda(\mathbf{r}^-) = |\mu^-|$. $\square$

3 Density, recurrence, and flows

This section establishes fundamentals of the Weil-Petersson geodesic flow on $\mathscr{M}^1(S)$, which, while standard for complete Riemannian manifolds of negative curvature, require more care due to the lack of completeness of the Weil-Petersson metric. In particular, the non-refraction of geodesics at the completion, Theorem 2.4 plays a crucial role in establishing that (1) the bi-infinite and recurrent geodesics each have full measure (Proposition 1.5 and 3.4), and (2) each asymptote class of infinite rays has a representative based at each $X \in \mathrm{Teich}(S)$ (Theorem 1.2).

In [Br2], the CAT(0) geometry of the Weil-Petersson completion and Theorem 2.4 are employed to show the following.

Theorem 3.1. ([Br2, Thm. 1.5]) *The finite rays are dense in the visual sphere.*

Wolpert observed that one obtains the following generalization (see [Wol5, Sec. 5]).

Theorem 3.2 (Wolpert)**.** *Restrictions to* $\mathrm{Teich}(S)$ *of Weil-Petersson geodesics in* $\overline{\mathrm{Teich}(S)}$ *joining pairs of maximally noded surfaces are dense in the unit tangent bundle* $T^1\mathrm{Teich}(S)$.

We recall a key element of the proof.

Lemma 3.3 (Wolpert)**.** *The finite rays have measure zero in the visual sphere.*

(See [Wol5, Wol6]).

Proof. Given a simplex σ in $\mathscr{C}(S)$, consider the natural geodesic retraction map from a given null-stratum $\mathscr{S}_\sigma$ onto the unit tangent sphere at $X \in \mathrm{Teich}(S)$, sending each point $Z \in \mathscr{S}_\sigma$ to the unit tangent at X in the direction of the unique geodesic from X to Z. Wolpert observes this map is Lipschitz from the intrinsic metric on $\mathscr{S}_\sigma$ to the standard metric on the unit tangent sphere. As each stratum has positive complex co-dimension, the image of $\overline{\mathrm{Teich}(S)} \setminus \mathrm{Teich}(S)$ has Hausdorff measure zero in the (real co-dimension 1) visual sphere. It follows that *infinite* directions have full measure. $\square$

Proposition 1.5 follows as an immediate corollary.

Proposition 1.5. *The geodesic flow is defined for all time on a full Liouville measure subset* $\mathscr{M}^1_\infty(S)$ *of* $\mathscr{M}^1(S)$*, consisting of bi-infinite geodesics.*

Proof. That the infinite rays have full-measure in the unit tangent bundle $T^1_X\mathrm{Teich}(S)$ at $X \in \mathrm{Teich}(S)$ implies that the directions determining bi-infinite geodesics have full measure in $T^1_X\mathrm{Teich}(S)$. By Fubini's theorem, the union over X of their projections determines a flow-invariant subset of $\mathscr{M}^1(S)$ of full measure. $\square$

A geodesic ray $\mathbf{r}$ based at $X \in \mathscr{M}(S)$ is *divergent* if for each compact set $K \subset \mathscr{M}(S)$, there is a T for which $\mathbf{r}(t) \cap K = \varnothing$ for each $t > T$. A ray $\mathbf{r}$ is called *recurrent* if it is not divergent.

Alternatively, Mumford's compactness theorem [Mum], guarantees that given $\varepsilon > 0$ the "ε-thick-part"

$$\mathrm{Teich}_{\geq\varepsilon}(S) = \{X \in \mathrm{Teich}(S) \mid \ell_\gamma(X) \geq \varepsilon,\ \gamma \in \mathcal{S}\}$$

of Teichmüller space projects to a compact subset of $\mathscr{M}(S)$. Thus we may characterize recurrent rays equivalently by the condition that there is an $\varepsilon > 0$ and a sequence of times $t_n \to \infty$ so that $\mathbf{r}(t_n) \subset \mathrm{Teich}_{\geq\varepsilon}(S)$.

A geodesic $\mathbf{g}$ is *doubly recurrent* if basepoint $X \in \mathbf{g}$ divides $\mathbf{g}$ into two recurrent rays based at X.

Taking Proposition 1.5 together with the Poincaré recurrence theorem, we have the following.

Proposition 3.4. *The recurrent rays and doubly recurrent geodesics in $\mathscr{M}^1(S)$ determine full-measure invariant subsets.*

Proof. The geodesic flow is volume-preserving on $\mathscr{M}^1(S)$, by Liouville's theorem (see [CFS, §2, Thm. 2]), and thus finiteness of the Weil-Petersson volume of $\mathscr{M}(S)$ ([Mas1, Wol2]), and hence of $\mathscr{M}^1(S)$, guarantees that no positive measure set of geodesics can be divergent by Poincaré recurrence. $\square$

The construction of an infinite ray at $Y \in \mathrm{Teich}(S)$ asymptotic to a given ray at $X \in \mathrm{Teich}(S)$ is an essential tool in our discussion. This is a general feature of complete CAT(0) spaces, as shown in [BH, II.8, 8.3], and thus applies to the completion $\overline{\mathrm{Teich}(S)}$. More care is required, however, to show that the resulting infinite ray in $\overline{\mathrm{Teich}(S)}$ actually determines an infinite ray in $\mathrm{Teich}(S)$. Indeed, the possibility that a limit of unbounded or even infinite geodesics might be finite cannot be ruled out a priori, as was shown in [Br2] (see also [Wol5]). This is also a consequence of Proposition 1.5. Theorem 1.2 follows from a key application of Theorem 2.4, the non-refraction of geodesics in the Weil-Petersson completion.

Theorem 1.2. (BOUNDARY AT INFINITY) *Let $X \in \mathrm{Teich}(S)$ be a basepoint.*

1. *For any $Y \in \mathrm{Teich}(S)$ with $Y \neq X$, and any infinite ray $\mathbf{r}$ based at X there is a unique infinite ray $\mathbf{r}'$ based at Y with $\mathbf{r}'(t) \in \mathrm{Teich}(S)$ for each t so that $\mathbf{r}'$ lies in the same asymptote class as $\mathbf{r}$.*

2. *The change of basepoint map restricts to a homeomorphism on the infinite rays.*

Remark. Because of totally geodesic flats in the completion arising from product strata, the condition that rays be merely asymptotic, namely, that they remain a bounded distance apart, cannot be improved to the condition that they be strongly asymptotic, though we will see this follows for recurrent rays (Theorem 4.1).

Proof. It is a general consequence of [BH, II.8, 8.3] applied to the complete CAT(0) space $\overline{\text{Teich}(S)}$ that we have a unique infinite geodesic ray $\mathbf{r}'(t)$ in $\overline{\text{Teich}(S)}$ based at Y in the asymptote class of $\mathbf{r}$ based at X. Indeed, the ray $\mathbf{r}'(t)$ is the limit of finite-length geodesics $\overline{Y\mathbf{r}(t)}$ joining Y to points along the ray $\mathbf{r}$ with their parametrizations by arclength, a fact we note for future reference.

It remains only to conclude that $\mathbf{r}'(t) \in \text{Teich}(S)$ for each $t > 0$. But by Theorem 2.4, for each $T > 0$ the geodesic $\mathbf{r}'([0,T])$ has interior $\mathbf{r}'((0,T))$ in the stratum $\mathscr{S}_{\sigma_0 \cap \sigma_T}$ where $\mathbf{r}'(0) \in \mathscr{S}_{\sigma_0}$ and $\mathbf{r}'(T) \in \mathscr{S}_{\sigma_T}$. But since $Y \in \text{Teich}(S)$ we have $\sigma_0 = \varnothing$, so $\mathbf{r}'(t)$ lies in the main stratum $\mathscr{S}_\varnothing = \text{Teich}(S)$ for each $t < T$. Since T is arbitrary, the conclusion follows.

It is general for a CAT(0) space that given a basepoint X, and an infinite ray $\mathbf{r}$ at X, the ray $\mathbf{r}$ is the unique representative of its asymptote class that is based at X. Thus, we have a unique infinite ray based at a fixed X in each asymptote class. Applying the CAT(0)-geometry of $\overline{\text{Teich}(S)}$, it follows that if $\mathbf{r}_n$ is a sequence of rays based at X with convergent initial tangents to the initial tangent of the infinite ray $\mathbf{r}_\infty$, then the corresponding infinite rays $\mathbf{r}'_n$ based at Y in the same asymptote class converge to the ray $\mathbf{r}'_\infty$ based at Y in the same asymptote class as $\mathbf{r}_\infty$. Thus the change of basepoint map is a homeomorphism on the infinite rays. $\square$

We remark that the assumption that Y lies in the interior of Teich(S) is just for simplicity: the same argument may be carried out to prove the following stronger statement.

Theorem 3.5. *Let σ and σ' be simplices in $\widehat{\mathscr{C}(S)}$. Let Y lie in the interior of a boundary stratum $\mathscr{S}_\sigma$. Then given an infinite ray $\mathbf{r}$ in $\overline{\text{Teich}(S)}$ based at $X \in \mathscr{S}_{\sigma'}$, there is a unique infinite ray $\mathbf{r}'$ based at Y with $\mathbf{r}'(t) \in \text{Teich}(S) \cup \mathscr{S}_\sigma$ for each t so that $\mathbf{r}'$ lies in the same asymptote class as $\mathbf{r}$.*

Proof. The proof goes through as before with the additional observation that for each s the limit $g_\infty([0,s))$ lies in $\text{Teich}(S) \cup \mathscr{S}_\sigma$ by Theorem 2.4. $\square$

4 Ending laminations and recurrent geodesics

The primary goal of this section is to establish Theorem 1.1.

Theorem 1.1. (Recurrent Ending Lamination Theorem) *Let* $\mathbf{r}$ *be a recurrent Weil-Petersson geodesic ray in* $\text{Teich}(S)$ *with ending lamination* $\lambda(\mathbf{r})$. *If* $\mathbf{r}'$ *is any other geodesic ray with ending lamination* $\lambda(\mathbf{r}') = \lambda(\mathbf{r})$ *then* $\mathbf{r}'$ *is strongly asymptotic to* $\mathbf{r}$.

The main technical tool in this section will be the following application of the Gauss-Bonnet theorem.

Theorem 4.1. *Let* $\mathbf{r}$ *be a recurrent Weil-Petersson geodesic ray. Then if* $\mathbf{r}'$ *is a ray asymptotic to* $\mathbf{r}$ *then* $\mathbf{r}$ *is strongly asymptotic to* $\mathbf{r}'$.

We wish to harness the fact that the recurrent ray $\mathbf{r}$ returns to a portion of $\mathscr{M}(S)$ where the sectional curvatures are definitely bounded away from 0. To do this we employ the technique of simplicial ruled surfaces, similar to Thurston's *pleated annulus* argument (cf. [Th2] and similar methods in [Can1] – see also [Bon1, Can2, Sou]).

For the purposes of the proof we make the following definition:

Definition 4.2. *Given a Weil-Petersson geodesic ray* $\mathbf{r}\colon [0,T] \to \text{Teich}(S)$ *parametrized by arclength, and an* $\varepsilon > 0$, *we say a* $t > 0$ *is an* ε-recurrence *for* $\mathbf{r}$ *if* $\mathbf{r}(t)$ *is* ε*-thick. Given* $\delta > 0$, *a collection* $\{t_k\}$ *of* ε*-recurrences for a ray* $\mathbf{r}$ *is* δ-separated *if* $|t_k - t_{k-1}| > \delta$.

Proof of Theorem 4.1. Let $\mathbf{r}$ and $\mathbf{r}'$ be asymptotic rays, and fix a parametrization of $\mathbf{r}$ for which $\mathbf{r}(t)$ is the nearest point projection on $\mathbf{r}$ from the point $\mathbf{r}'(t)$, where $\mathbf{r}'(t)$ is parametrized by arclength.

Given points X, Y, and Z in $\text{Teich}(S)$, let $\triangle(XYZ)$ denote the ruled triangle with vertices X, Y, and Z ruled by geodesics joining Z to points along $\overline{XY}$. Then given $T > 0$, we consider the ruled triangle

$$\Delta_T = \triangle(\mathbf{r}(0)\mathbf{r}(T)\mathbf{r}'(0)).$$

The Weil-Petersson Riemannian metric induces a smooth metric σ_T on Δ_T whose Gauss curvature is pointwise bounded from above by the upper bound on the ambient sectional curvatures.

By recurrence of $\mathbf{r}$, there is an $\varepsilon > 0$ so that for any $\delta > 0$ there is an infinite collection δ-separated ε-recurrences for $\mathbf{r}$. Since the ε-thick part projects to a compact subset of $\mathscr{M}(S)$ [Mum], it follows that there is a $\delta_0 > 0$ so that if t is an ε-recurrence for $\mathbf{r}$ we have

$$\mathscr{N}_{\delta_0}(\mathbf{r}(t)) \subset \text{Teich}_{\geq \varepsilon/2}(S) \tag{4.1}$$

Thus, the sectional curvatures on $\mathscr{N}_{\delta_0}(\mathbf{r}(t))$ are bounded from above by a negative constant $\kappa_{\varepsilon/2} < 0$, and thus the Gaussian curvature of σ_T on

$$\Delta_T \cap \mathscr{N}_{\delta_0}(\mathbf{r}(t))$$

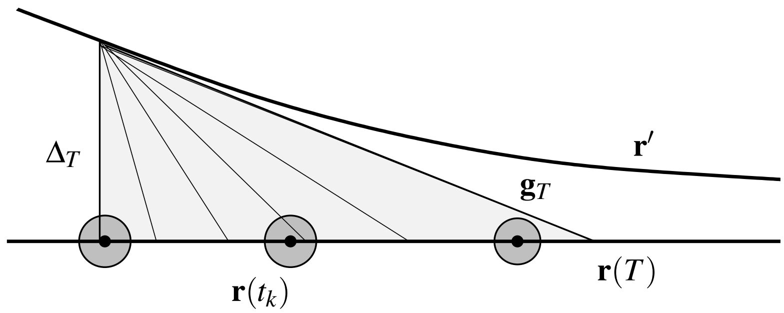

Figure 1. The ruled triangle Δ_T.

is bounded above by $\kappa_{\varepsilon/2}$ as well.

Assume there is a $\delta \in (0, \delta_0)$ so that the distance from $\mathbf{r}(t)$ to $\mathbf{r}'(t)$ is bounded below by δ. The segments $\mathbf{g}_T = \overline{\mathbf{r}'(0)\,\mathbf{r}(T)}$ converge to the infinite ray $\mathbf{r}'$, as $T \to \infty$, so for fixed t we have $\mathbf{g}_T(t) \to \mathbf{r}'(t)$ as $T \to \infty$. It follows that for each $t > 0$ for which $\mathbf{r}(t) \in \mathrm{Teich}_{\geq \varepsilon}(S)$, there is a $T > t$ so that $\mathbf{g}_T(t)$ has distance at least $\delta/2$ from $\mathbf{r}(t)$.

We note that the inclusion map on Δ_T is 1-Lipschitz from the σ_T-metric to the Weil-Petersson metric. Then the intersection of the triangle Δ_T with the neighborhood $\mathcal{N}_{\delta/2}(\mathbf{r}(t))$ contains a region with area at least $\pi\delta^2/16$ in the intrinsic metric on Δ_T, since each such intersection contains a sector in Δ_T with radius $\delta/2$ and angle $\pi/2$ whose σ_T-area is minorized by the area of a Euclidean sector with the same radius and angle.

Let $\{t_k\}_k$ be an infinite collection of δ-separated ε-recurrences for $\mathbf{r}$. Given $N > 0$ take T_N so that $\mathbf{g}_{T_N}(t_k)$ has distance at least $\delta/2$ from $\mathbf{r}(t_k)$ for each $k \leq N$. Then we have the estimate

$$\left| \int_{\Delta_{T_N}} \kappa \, dA \right| > N \left| \kappa_{\varepsilon/2} \right| \frac{\pi\delta^2}{16} \tag{4.2}$$

on the absolute value of the integral of the Gauss curvature κ over Δ_{T_N}.

The Gauss-Bonnet Theorem bounds the integral (4.2) from above by π, independent of T_N, so letting

$$N(\varepsilon, \delta) = \frac{16}{\left| \kappa_{\varepsilon/2} \right| \delta^2} \tag{4.3}$$

the bound $N < N(\varepsilon, \delta)$ to the cardinality of the set $\{t_k\}_{k=0}^N$ follows.

Since N is arbitrary, (4.3) contradicts the recurrence of $\mathbf{r}$ to the ε-thick part, and we conclude the existence of $\hat{t}$ for which $\mathbf{g}_T(\hat{t}) \cap \mathcal{N}_{\delta/2}(\mathbf{r}(\hat{t})) \neq \emptyset$ independent of T. Since we have $\mathbf{g}_T(\hat{t}) \to \mathbf{r}'(\hat{t})$ as $T \to \infty$, we conclude that $\mathbf{r}'(\hat{t}) \cap \mathcal{N}_{\delta}(\mathbf{r}(\hat{t})) \neq \emptyset$. Since $\delta > 0$ is arbitrary, and since the distance from $\mathbf{r}(t)$ to $\mathbf{r}'(t)$ is a non-increasing function in a CAT(0) space, it follows that the rays are strongly asymptotic. $\square$

Remark: M. Bestvina and K. Fujiwara have observed indepenently the applicability of this ruled surface technique to the study of action of $\mathrm{Mod}(S)$ on $\overline{\mathrm{Teich}(S)}$ as the isometry group of a CAT(0)-space (cf. [BeFu]).

We employ the fact that recurrent rays exhibit such strongly asymptotic behavior to conclude Theorem 1.3.

Theorem 1.3. (RECURRENT VISIBILITY) *Let $\mathbf{r}^+$ and $\mathbf{r}^-$ be two distinct infinite rays based at X.*

1. *If $\mathbf{r}^+$ is recurrent, then there is a single bi-infinite geodesic $\mathbf{g}(t)$ so that $\mathbf{g}^+ = \mathbf{g}|_{[0,\infty)}$ is* strongly *asymptotic to $\mathbf{r}^+$ and $\mathbf{g}^- = \mathbf{g}|_{(-\infty,0]}$ is asymptotic to $\mathbf{r}^-$. In particular, if both $\mathbf{r}^+$ and $\mathbf{r}^-$ are recurrent, then $\mathbf{g}$ is strongly asymptotic to both $\mathbf{r}^-$ and $\mathbf{r}^+$.*

2. *If μ in the measured lamination space $\mathscr{ML}(S)$ has bounded length on $\mathbf{r}^\pm$ then it has bounded length on $\mathbf{g}^\pm$.*

Proof. We seek to exhibit a bi-infinite Weil-Petersson geodesic $\mathbf{g}\colon \mathbb{R} \to \mathrm{Teich}(S)$ with the property that $\mathbf{g}$ is strongly asymptotic to the recurrent ray $\mathbf{r}^+$ in positive time and asymptotic to $\mathbf{r}^-$ in negative time. In other words, we claim there is a reparametrization $t \mapsto s(t) > 0$ so that we have

$$d(\mathbf{g}(s(t)), \mathbf{r}^+(t)) \to 0$$

as $t \to \infty$, and

$$d(\mathbf{g}(t), \mathbf{r}^-(-t))$$

is bounded for $t < 0$.

Consider geodesic chords $\mathbf{g}_n$ joining $\mathbf{r}^+(n)$ to $\mathbf{r}^-(n)$ for integers $n > 0$. We wish to show that after passing to a subsequence there are paramater values $\hat{t}_n$ so that $\mathbf{g}_n(\hat{t}_n)$ converges to some $Z \in \mathrm{Teich}(S)$.

Assume $\mathbf{r}^+$ is recurrent to the ε-thick part. For this ε, let δ_0 be chosen as in (4.1). Then given $\delta \in (0, \delta_0)$ we let $\{t_k\}_{k=0}^\infty$ be δ-separated ε-recurrences for $\mathbf{r}^+$.

Since the rays $\mathbf{r}^-$ and $\mathbf{r}^+$ are assumed distinct, we may omit finitely many recurrences and assume that for each k the distance of $\mathbf{r}^+(t_k)$ from $\mathbf{r}^-$ is at least δ. Otherwise, the two rays lie in the same asymptote class, and are thus identical since they are based at the same point in a CAT(0) space.

We claim that there is a $\hat{k}$ so that

$$\mathbf{g}_n \cap \mathscr{N}_{\delta/2}(\mathbf{r}^+(t_{\hat{k}})) \neq \emptyset$$

for all n. Let $s_0 > 0$ be the minimal parameter so that $\mathbf{r}^-(s_0)$ has distance at least δ from $\mathbf{r}^+$. Then for $n > s_0$, consider the ruled triangle in $\mathrm{Teich}(S)$ given by

$$\mathscr{T}_n = \triangle(\mathbf{r}^+(0)\mathbf{r}^+(n)\mathbf{r}^-(n)).$$

If N is chosen so that $t_N < n$, and

$$\mathscr{N}_{\delta/2}(\mathbf{r}^+(t_N)) \cap \mathbf{g}_n = \emptyset,$$

then as in (4.3) we have the upper bound

$$N < N(\varepsilon, \delta).$$

Since $\{t_k\}_k$ is infinite, we conclude there is a $\hat{k}$ so that $\mathbf{g}_n$ intersects $\mathscr{N}_{\delta/2}(t_{\hat{k}})$ for all n sufficiently large. As the $\delta/2$-ball $\mathscr{N}_{\delta/2}(\mathbf{r}^+(t_{\hat{k}}))$ lies in $\mathrm{Teich}_{\geq\varepsilon/2}(S)$, it is precompact, and we may pass to a convergent subsequence so that points $Z_n \in \mathbf{g}_n$ converge to a limit Z in $\mathrm{Teich}(S)$.

Reparametrizing $\mathbf{g}_n$ by arclength so that $\mathbf{g}_n(0) = Z_n$, we observe that for each $t \in \mathbb{R}^+$ the sequence $\{\mathbf{g}_n(t)\}_n$ is Cauchy. To see this, note that if $\mathbf{h}_n^+$ is the segment joining Z to $\mathbf{r}^+(n)$ parametrized by arclength, choosing n_t so that so that $d(Z, \mathbf{r}^+(n)) > t$ for each $n \geq n_t$, the sequence $\{\mathbf{h}_n^+(t)\}_{n=n_t}^\infty$ is Cauchy (as in [BH, II.8,8.3] and its use in Theorem 1.2). The assertion then follows from the observation that $\mathrm{CAT}(0)$ geometry guarantees

$$d(\mathbf{h}_n^+(t), \mathbf{g}_n^+(t)) < d(Z_n, Z) \to 0.$$

The symmetric argument shows $\{\mathbf{g}_n(-t)\}_n$ is Cauchy as well.

There is thus a limiting geodesic $\mathbf{g}\colon \mathbb{R} \to \overline{\mathrm{Teich}(S)}$ for which the ray $\mathbf{g}^+ = \mathbf{g}|_{[0,\infty)}$ lies in the asymptote class of $\mathbf{r}^+$ and $\mathbf{g}^- = \mathbf{g}|_{(-\infty,0]}$ lies in the asymptote class of $\mathbf{r}^-$. Since $\mathbf{g}^+$ and $\mathbf{g}^-$ are the unique representatives of the asymptote classes of $\mathbf{r}^+$ and $\mathbf{r}^-$ based at Z, an application of Theorem 1.2 ensures $\mathbf{g}^+$ and $\mathbf{g}^-$ lie entirely within $\mathrm{Teich}(S)$ as claimed.

For statement (2), we note that by Theorem 1.2, the ray $\mathbf{g}^+$ is the limit of geodesics $\mathbf{g}_n^+$ joining $\mathbf{g}(0) = Z$ to $\mathbf{r}^+(n)$, so if μ has bounded length along $\mathbf{r}^+$, then convexity of the length of μ guarantees that the length of μ is uniformly bounded on $\mathbf{g}^+$. The same argument applies to $\mathbf{g}^-$. Statement (2) follows. $\square$

In Section 2, we employed the boundedness of length functions for ending measures along a ray to establish that the ending lamination is well defined. For a recurrent ray, however, we can guarantee that the length of any lamination with bounded length decays to zero.

Lemma 4.3. *Let $\mathbf{r}(t)$ be a recurrent ray, and let $\mu \in \mathscr{ML}(S)$ be any lamination with $\ell_\mu(\mathbf{r}(t)) < K$ along $\mathbf{r}(t)$. Then we have*

$$\ell_\mu(\mathbf{r}(t)) \to 0$$

as $t \to \infty$.

Proof. Assume $\mathbf{r}(t)$ recurs to the ε-thick part at times $t_n \to \infty$. Wolpert's extension of his convexity theorem for geodesic length functions guarantees that the length of $\mu \in \mathscr{ML}(S)$, in addition to being convex along geodesics [Wol4], satisfies the following stronger convexity property: given $\varepsilon > 0$, there is a $c > 0$ so that at each t for which $\mathbf{r}(t)$ lies in the ε-thick part, we have

$$\ell''_\mu(\mathbf{r}(t)) > c\ell_\mu(\mathbf{r}(t))$$

(see [Wol6]). The proof of the Lemma then follows from the observation that if the bounded convex function $\ell_\mu(\mathbf{r}(t))$ does not tend to zero, then we nevertheless have $\ell_\mu(\mathbf{r}(t)) \to C > 0$ as $t \to \infty$, which guarantees that $\ell''_\mu(\mathbf{r}(t)) \to 0$ by convexity. This contradicts the above inequality at the times t_n for n sufficiently large. $\square$

Though the ending lamination need not fill the surface in general, the recurrent rays provide a class of rays where each lamination with bounded length along the ray fills S.

Proposition 4.4. *Let μ be any measured lamination with bounded length along the recurrent ray $\mathbf{r}(t)$. Then μ is a filling lamination.*

Proof. Assume μ does not fill, and let $S(\mu)$ be the supporting subsurface for its support $|\mu|$. Let $\gamma_n \in \mathscr{C}(S(\mu))$ be a sequence of simple closed curves whose projective classes $[\gamma_n]$ converge to $[\mu]$ in $\mathscr{PML}(S)$. Note in particular that

$$i(\partial S(\mu), \gamma_n) = 0$$

for each n.

We claim that given any $Z \in \text{Teich}(S(\mu))$ there is a Weil-Petersson ray $\hat{\mathbf{r}}$ in $\text{Teich}(S(\mu))$ based at Z along which μ has bounded length. To see this, note that any limit $\hat{\mathbf{r}}$ of finite-length rays $\overline{Z\,Z_n}$ joining Z to nodal surfaces Z_n in $\overline{\mathscr{S}_{\gamma_n}} \cap \overline{\text{Teich}(S(\mu))}$ has the property that $[\mu]$ is the projective class of a lamination with bounded length along $\hat{\mathbf{r}}$ by Lemma 2.10. The fact that μ fills $S(\mu)$ guarantees that $\hat{\mathbf{r}}$ has no pinching curves. Thus $\hat{\mathbf{r}}$ has infinite length.

Letting $\sigma_\mu \in \widehat{\mathscr{C}(S)}$ be the simplex spanned by the curves in $\partial S(\mu)$ we note that the stratum $\mathscr{S}_{\sigma_\mu}$ is the metric product of Weil-Petersson metrics on $\text{Teich}(S(\mu))$ and the Weil-Petersson metrics on $\text{Teich}(Y)$ where Y is the disjoint union of non-annular components of $S \setminus S(\mu)$.

Together with the basepoint X, then, the ray $\hat{\mathbf{r}}$ naturally determines a ray $\bar{\mathbf{r}}$ in the stratum $\mathscr{S}_{\sigma_\mu}$ by taking the projection of $\overline{\mathbf{r}(t)}$ to $\mathrm{Teich}(S(\mu))$ to be $\hat{\mathbf{r}}(t)$ and identifying each other coordinate of $\bar{\mathbf{r}}(t)$ in the product decomposition of $\mathscr{S}_{\sigma_\mu}$ with the (constant) coordinate function of the nearest point projection of X to $\mathscr{S}_{\sigma_\mu}$.

Applying Theorem 3.5, there is a unique ray $\mathbf{r}'$ based at X asymptotic to $\bar{\mathbf{r}}$. The ray $\mathbf{r}'$ is constructed as a limit of segments $\mathbf{g}_t = \overline{X\,\bar{\mathbf{r}}(t)}$ joining X to points along $\bar{\mathbf{r}}$. The length of μ and each curve $\gamma \subset \partial S(\mu)$ is uniformly bounded on the segments $\overline{X\,\bar{\mathbf{r}}(t)}$, by convexity of length functions. Applying continuity of length, then, we have a $K > 0$ so that

$$\ell_\mu(\mathbf{r}'(t)) < K \quad \text{and} \quad \ell_\gamma(\mathbf{r}'(t)) < K$$

for each $\gamma \in \partial S(\mu)$.

If $\mathbf{r}'$ is distinct from $\mathbf{r}$, however, Theorem 1.3 guarantees that we may find a bi-infinite geodesic $\mathbf{g}$ whose forward trajectory is strongly asymptotic to $\mathbf{r}$, by recurrence, and so that $\mathbf{g}|_{(-\infty,0]}$ stays a bounded distance from $\mathbf{r}'$. Once again, the length $\ell_\mu(\mathbf{g}(t))$ of μ is uniformly bounded over the entire bi-infinite geodesic $\mathbf{g}$. Since $\ell_\mu(\mathbf{g}(t))$ approaches 0 as $t \to \infty$ by Lemma 4.3, the boundedness of $\ell_\mu(\mathbf{g}(t))$ yields a contradiction to its strict convexity. We conclude that $\mathbf{r} = \mathbf{r}'$ and thus that γ has bounded length along the ray $\mathbf{r}(t)$. But applying Lemma 4.3 once again, boundedness implies that the length of γ tends to zero along $\mathbf{r}(t)$, violating recurrence of $\mathbf{r}(t)$.

We conclude that μ fills S. $\square$

Theorem 1.1 will be a direct consequence of the following characterization of measures that arise with bounded length along a recurrent Weil-Petersson ray, together with Theorem 1.3.

Lemma 4.5. *Let* $\mathbf{r}$ *be a recurrent ray with ending lamination* $\lambda = \lambda(\mathbf{r})$*. If* $\mu \in \mathscr{ML}(S)$ *then the following are equivalent:*

1. *The lamination* μ *has support* λ*.*

2. *The length function* $\ell_\mu(\mathbf{r}(t))$ *is bounded.*

3. $\ell_\mu(\mathbf{r}(t)) \to 0$.

Proof. We first verify that (1) implies (2). Let Σ denote the simplex of projective classes of measures on λ in $\mathscr{PML}(S)$, and let $\hat{\mu} \in \mathscr{ML}(S)$ be a representative of the projective class determined by a point in the interior of the top dimensional face. Then $\hat{\mu}$ is a positive linear combination of all ergodic measures on λ.

Let γ_n be a sequence of simple closed curves for which the projective classes $[\gamma_n]$ converge to $[\hat{\mu}]$, and let $\mathbf{r}_n$ be a sequence of finite rays based at X limiting to points Z_n in the strata $\mathscr{S}_{\gamma_n}$. Since γ_n are pinching curves for $\mathbf{r}_n$, Lemma 2.10 guarantees that any limit $\mathbf{r}_\infty$ of a convergent subsequence of $\mathbf{r}_n$ has the property that $\hat{\mu}$ has bounded length along $\mathbf{r}_\infty$. Since $\hat{\mu}$ is a positive linear combination of all the ergodic measures on λ, it follows that each ergodic measure on λ has bounded length along $\mathbf{r}_\infty$. Hence, any measured lamination representing a projective class in Σ has bounded length along $\mathbf{r}_\infty$ since each is a linear combination of ergodic measures.

Since $\lambda(\mathbf{r})$ is filling, by Proposition 4.4, we have that $\hat{\mu}$ is filling. This guarantees that $\mathbf{r}_\infty$ has infinite length, since otherwise $\mathbf{r}_\infty$ would have a pinching curve γ with $i(\gamma, \hat{\mu}) > 0$, violating the length bound on $\hat{\mu}$ along $\mathbf{r}_\infty$.

If $\bar{\mu}$ is any ending measure for $\mathbf{r}$, then $\bar{\mu}$ represents a projective class in Σ, and thus has bounded length along $\mathbf{r}_\infty$. If $\mathbf{r}$ and $\mathbf{r}_\infty$ are distinct rays, then Theorem 1.3 guarantees that we have a bi-infinite geodesic $\mathbf{g}(t)$ asymptotic to $\mathbf{r}$ and $\mathbf{r}_\infty$ along which $\bar{\mu}$ has bounded length, which contradicts strict convexity of the length function for $\bar{\mu}$ along $\mathbf{g}$. It follows that $\mathbf{r} = \mathbf{r}_\infty$.

Since μ also represents a measure in Σ, it follows that μ has bounded length along $\mathbf{r}$, verifying that (1) implies (2).

Conclusion (3) follows from conclusion (2) by an application of Lemma 4.3.

By Corollary 2.12, (3) implies that any ending measure μ' for $\mathbf{r}$ has the property that

$$i(\mu, \mu') = 0.$$

By Proposition 4.4 each of μ and μ' fills, so they have identical support, verifying conclusion (1), and hence proving the Lemma. $\square$

We are now ready to prove Theorem 1.1, that recurrent rays with the same ending lamination are strongly asymptotic.

Proof of Theorem 1.1. Let $\mathbf{r}$ be a recurrent ray based at $X \in \mathrm{Teich}(S)$, with ending lamination $\lambda = \lambda(\mathbf{r})$. Let $\mathbf{r}'$ be another ray based at $X \in \mathrm{Teich}(S)$ with ending lamination λ, and let μ be an ending measure for $\mathbf{r}'$. Then μ has support λ, so by an application of Lemma 4.5 μ has bounded length along $\mathbf{r}$ as well.

If $\mathbf{r}$ and $\mathbf{r}'$ are distinct rays, then Theorem 1.3 guarantees that we have a bi-infinite geodesic $\mathbf{g}(t)$ asymptotic to $\mathbf{r}$ and $\mathbf{r}'$ along which μ has bounded length, which contradicts strict convexity of the length function for μ along $\mathbf{g}$. It follows that $\mathbf{r} = \mathbf{r}'$.

If $\mathbf{r}''$ is a ray based at $Y \neq X$, with ending lamination λ, there is a unique ray $\mathbf{r}'$ based at X in the asymptote class of $\mathbf{r}''$ by Theorem 1.2. Applying the above argument to $\mathbf{r}'$ we may conclude that $\mathbf{r}''$ and $\mathbf{r}$ are in the same asymptote class. Theorem 4.1 then guarantees that $\mathbf{r}''$ and $\mathbf{r}$ are strongly asymptotic, concluding the proof. $\square$

As a further consequence, we note the following.

Corollary 4.6. *Let $\mathbf{g}(t)$ be a bi-infinite Weil-Petersson geodesic whose forward trajectory is recurrent. Then the ending laminations λ^+ and λ^- for the rays $\mathbf{g}^+ = \{\mathbf{g}(t)\}_{t=0}^{\infty}$ and $\mathbf{g}^- = \{\mathbf{g}(t)\}_{t=0}^{-\infty}$ bind the surface S.*

Proof. The ending lamination λ^+ for the forward trajectory fills the surface, so the ending lamination for the backward trajectory must intersect it, since otherwise the laminations λ^- and λ^+ would be identical therefore we would have $\mathbf{g}^+ = \mathbf{g}^-$ by Theorem 1.1, a contradiction. $\square$

To derive Corollary 1.4, we establish a final further continuity property for ending measures when the limit is recurrent.

Proposition 4.7. *If $\mathbf{r}_n$ is a convergent sequence of rays at X with a recurrent limit $\mathbf{r}$, any sequence μ_n of ending measures or pinching curves for $\mathbf{r}_n$ converges in $\mathscr{PML}(S)$ up to subsequence to a measure on $\lambda(\mathbf{r})$.*

Proof. Let μ be any limit of μ_n in $\mathscr{PML}(S)$ after passing to a subsequence. Then by Lemma 2.10, the length $\ell_\mu(\mathbf{r}(t))$ is bounded. Since $\mathbf{r}$ is recurrent, any ending measure μ' for $\mathbf{r}$ fills S by Proposition 4.4. But by Corollary 2.12, we have

$$i(\mu, \mu') = 0$$

so μ and μ' have identical support since μ' is filling. Hence, μ is a measure on $\lambda(\mathbf{r})$. $\square$

Restricting to the recurrent rays, we obtain Corollary 1.4.

Corollary 1.4. *The map λ that associates to an equivalence class of recurrent rays its ending lamination is a homeomorphism to the subset $\mathscr{REL}(S)$ in $\mathscr{EL}(S)$.*

Proof. That the map is a bijection follows from the fact that $\mathscr{REL}(S)$ is defined as its image and from Theorem 1.1.

To show continuity in each direction, we begin by noting that although the topology induced by forgetting the measure on a measured lamination is not a Hausdorff topology on the geodesic laminations admitting measures, it is Hausdorff when one restricts to those that fill the surface, namely, the subset $\mathscr{EL}(S)$ (see [Kla, §7]). As such it suffices to consider sequential limits to establish continuity.

Furthermore, the topology of convergence of asymptote classes of of infinite rays in $\partial_\infty \overline{\mathrm{Teich}(S)}$ agrees with the topology of convergence in the visual sphere of representatives emanating from a given basepoint X as in Theorem 1.2 (see also [BH, II.8,8.8]).

Let $\mathbf{r}_n$ be a sequence of recurrent rays based at X with recurrent limit $\mathbf{r}$ based at X. By Proposition 4.4, their ending laminations λ_n are filling laminations and thus determine points in $\mathscr{REL}(S)$. Their recurrent limit $\mathbf{r}$ has ending lamination $\lambda(\mathbf{r})$, with support identified with the support of a limit of measures on λ_n by Proposition 4.7, so λ is the limit of λ_n in $\mathscr{REL}(S)$, by the definition of the topology on $\mathscr{EL}(S)$.

For continuity in the other direction, compactness of the visual sphere at X guarantees that any sequence of laminations λ_n converging to λ_∞ in $\mathscr{REL}(S)$ determine a sequence of rays $\mathbf{r}_n$ at X with limit $\mathbf{r}_\infty$ at X after passing to a subsequence. A convergent family of measures μ_n on λ_n has limit μ_∞, a measure on λ_∞, with bounded length on the limiting ray $\mathbf{r}_\infty$ by Lemma 2.10. Since μ_∞ is filling, and any ending measure or weighted pinching curve μ' for $\mathbf{r}_\infty$ satisfies $i(\mu,\mu') = 0$, we conclude that μ' has the same support as μ_∞, namely λ_∞. Thus $\mathbf{r}_\infty$ is the recurrent ray at X determined (uniquely) by λ_∞. Since any accumulation point of the rays $\mathbf{r}_n$ has this property, the original sequence of rays itself was convergent to $\mathbf{r}_\infty$, obviating passage to subsequences. $\square$

5 The topological dynamics of the geodesic flow

We now relate the preceding results to the study of the Weil-Petersson geodesic flow on $\mathscr{M}^1(S)$.

Though it is seen in [Br2] that the change of basepoint map is discontinuous on the visual sphere, the visibility property for recurrent rays (Theorem 1.3) is sufficient to remedy the situation for considerations of topological dynamics, yielding Theorem 1.6, whose proof we now supply.

Theorem 1.6. (CLOSED ORBITS DENSE) *The set of closed Weil-Petersson geodesics is dense in $\mathscr{M}^1(S)$.*

Proof. Because of the density of doubly recurrent geodesics in the unit tangent bundle $T^1\mathrm{Teich}(S)$, it suffices by a diagonal argument to approximate a bi-recurrent direction with periodic geodesics.

Let $\{\mathbf{g}(t)\}_{t=-\infty}^{\infty}$ be a bi-infinite geodesic that is doubly recurrent. Let $X = \mathbf{g}(0)$ be a basepoint, and let λ^+ be the ending lamination for the forward ray $\mathbf{g}^+(t) = \{\mathbf{g}(t)\}_{t=0}^{\infty}$ and likewise let λ^- denote the ending lamination for the backward ray $\mathbf{g}^- = \{\mathbf{g}(-t)\}_{t=0}^{\infty}$. By Corollary 4.6, λ^+ and λ^- bind the surface S, so letting μ^+ and μ^- be measures on λ^+ and λ^-, respectively, any pair of simple closed curves γ^+ and γ^- very close to μ^+ and μ^- in $\mathscr{PML}(S)$ also bind S.

Letting τ_+ be a Dehn twist about γ^+ and τ_- be a Dehn twist about γ^-, the composition

$$\psi_k = \tau_+^k \circ \tau_-^k$$

is pseudo-Anosov for all k sufficiently large [Th3]. Furthermore, the stable and unstable laminations for ψ_k converge to γ^+ and γ^- in $\mathscr{PML}(S)$ as $k \to \infty$. Diagonalizing, then, we obtain a sequence of pseudo-Anosov mapping classes φ_n whose unstable and stable laminations μ_n^+ and μ_n^- converge to μ^+ and μ^- in $\mathscr{PML}(S)$. Since the supports $|\mu_n^\pm|$ and $|\mu^\pm|$ lie in $\mathscr{REL}(S)$, we have convergence of $|\mu_n^\pm|$ to $\lambda^\pm$ in $\mathscr{REL}(S)$ by the definition of the topology on $\mathscr{EL}(S)$.

Letting A_n be the axis for φ_n, we claim A_n is arbitrarily close to $\mathbf{g}$ at $\mathbf{g}(0)$ in the unit tangent bundle for n sufficiently large.

To see this, we apply Theorem 1.2 to obtain a ray $\mathbf{r}_n^+$ in $\mathscr{V}_X(S)$ asymptotic to A_n in the forward direction. We note that, as A_n is itself doubly recurrent, the ray $\mathbf{r}_n^+$ is strongly asymptotic to A_n, by Theorem 4.1, and that the ending lamination λ_n^+ for $\mathbf{r}_n^+$ is equal to the support of μ_n^+. It follows that λ_n^+ converges to λ^+ in $\mathscr{REL}(S)$. Likewise, if $\mathbf{r}_n^-$ denotes the ray in $\mathscr{V}_X(S)$ asymptotic to A_n in the negative direction, then $\lambda_n^- = \lambda(\mathbf{r}_n^-)$ converges to λ^- in $\mathscr{REL}(S)$. The parametrization of recurrent rays by their ending laminations in $\mathscr{EL}(S)$, Corollary 1.4, guarantees that $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ converge to $\mathbf{g}^+$ and $\mathbf{g}^-$ respectively.

Let $\varepsilon > 0$ be taken so that $\mathbf{g}^+$ and $\mathbf{g}^-$ recur to $\mathrm{Teich}_{\geq 4\varepsilon}(S)$. Assume $\delta_0 > 0$ is chosen as in (4.1) so that given $Z \in \mathrm{Teich}_{\geq\varepsilon}(S)$ and $\delta < \delta_0$ the δ-neighborhood $\mathscr{N}_\delta(Z)$ is precompact in $\mathrm{Teich}(S)$. Let $\{t_k\}_k$ and $\{s_k\}_k$ be 4δ-separated 4ε-recurrences for $\mathbf{g}^+$ and $\mathbf{g}^-$ respectively.

The convergence of $\mathbf{r}_n^+ \to \mathbf{g}^+$ and $\mathbf{r}_n^- \to \mathbf{g}^-$ guarantees that given any positive integer N, there is a n_N so that for $n > n_N$ the parameters $\{t_k\}_{k=0}^N$ and $\{s_k\}_{k=0}^N$ are 2δ-separated 2ε-recurrences for $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ respectively. Letting

$$\mathbf{g}_{n,T}^+ = \overline{X\, Z_n^+(T)} \quad \text{and} \quad \mathbf{g}_{n,T}^- = \overline{X\, Z_n^-(T)}$$

be the geodesic segments joining X to nearest points $Z_n^+(T)$ and $Z_n^-(T)$ on A_n to $\mathbf{r}_n^+(T)$ and $\mathbf{r}_n^-(T)$, we have the convergence of $\mathbf{g}_{n,T}^+$ to $\mathbf{r}_n^+$ and $\mathbf{g}_{n,T}^-$ to $\mathbf{r}_n^-$ as $T \to \infty$ for fixed n. There is a T_N, then, so that t_k and s_k are δ-separated ε-recurrences for $\mathbf{g}_{n,T_N}^+$ and $\mathbf{g}_{n,T_N}^-$ for each $k \leq N$ and $n > n_N$.

We wish to apply the ruled triangle argument of Theorem 4.1 to the two triangles $\Delta_n^+(T)$ and $\Delta_n^-(T)$ where

$$\Delta_n^+(T) = \triangle(Z_n^+(T)\, Z_n^0\, X) \quad \text{and} \quad \Delta_n^-(T) = \triangle(Z_n^-(T)\, Z_n^0\, X)$$

(see figure 2).

Taking $\delta < \delta_0$ and letting $N(\varepsilon,\delta)$ be as in the application of Gauss-Bonnet in (4.3) assume $n > n_{N(\varepsilon,\delta)}$ and $T > T_{N(\varepsilon,\delta)}$. If N' is the maximal integer such that

$$\cup_{k=1}^{N'} \left(\mathscr{N}_{\delta/2}(\mathbf{g}_{n,T}^+(t_k))\right) \cap A_n = \emptyset \quad \text{or} \quad \cup_{k=1}^{N'} \left(\mathscr{N}_{\delta/2}(\mathbf{g}_{n,T}^-(s_k))\right) \cap A_n = \emptyset$$

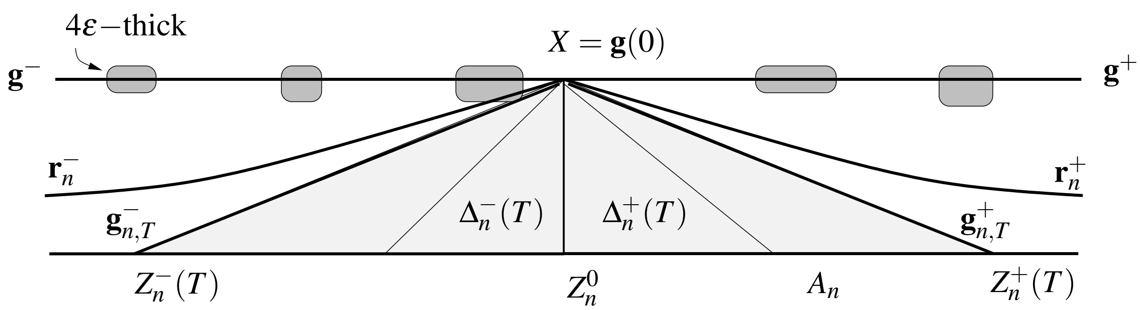

Figure 2. The axis A_n converges to $\mathbf{g}$.

it follows that

$$N' < N(\varepsilon, \delta).$$

Taking $\hat{k} \geq N(\varepsilon, \delta)$, then, we may conclude that for n and T sufficiently large we have

$$\mathscr{N}_{\delta/2}(\mathbf{g}^+_{n,T}(t_{\hat{k}})) \cap A_n \neq \emptyset \quad \text{and} \quad \mathscr{N}_{\delta/2}(\mathbf{g}^-_{n,T}(s_{\hat{k}})) \cap A_n \neq \emptyset.$$

Since for fixed n we have $\mathbf{g}^+_{n,T}(t_{\hat{k}}) \to \mathbf{r}^+_n(t_{\hat{k}})$ and $\mathbf{g}^-_{n,T}(t_{\hat{k}}) \to \mathbf{r}^-_n(t_{\hat{k}})$ as $T \to \infty$, it follows that

$$\mathscr{N}_{\delta}(\mathbf{r}^+_n(t_{\hat{k}})) \cap A_n \neq \emptyset \quad \text{and} \quad \mathscr{N}_{\delta}(\mathbf{r}^-_n(s_{\hat{k}})) \cap A_n \neq \emptyset$$

for n sufficiently large.

Properties of CAT(0) geometry guarantee that the points $Z^+_n(t_{\hat{k}})$ and $Z^-_n(s_{\hat{k}})$ on A_n bound geodesic subsegments $\ell_{n,\hat{k}}$ of A_n that converge up to subsequence to a geodesic segment $\ell_{\hat{k}}$ within δ of $\mathbf{g}$ as $n \to \infty$. Thus, a diagonal argument allows us to extract a bi-infinite geodesic limit A_∞ of the axes A_n so that for each $\delta > 0$, and $T' > 0$, A_∞ contains a subsegment of width at least T' within δ of $\mathbf{g}$. We conclude $A_\infty = \mathbf{g}$.

The projections of A_n to $\mathscr{M}(S)$ are closed geodesics approximating the doubly recurrent projection of $\mathbf{g}$ to $\mathscr{M}(S)$, as was desired. $\square$

Using the boundary theory for the recurrent rays and its connection with measured laminations, we can harness the north-south dynamics of pseudo-Anosov elements on $\mathscr{PML}(S)$ to establish Theorem 1.7 as a consequence of Theorem 1.6.

Theorem 1.7. (DENSE GEODESICS) *Given any $X \in$ Teich(S), there is a Weil-Petersson geodesic ray based at X whose projection to $\mathscr{M}^1(S)$ is dense.*

Proof. Given a pseudo-Anosov mapping class ψ let μ^+ and μ^- denote representatives of the attracting and repelling fixed points $[\mu^+]$ and $[\mu^-]$ for the action of ψ on $\mathscr{PML}(S)$, respectively. Let A_ψ denote its axis in the Weil-Petersson metric, and let $\lambda^+ = |\mu^+|$ and $\lambda^- = |\mu^-|$ denote the support in $\mathscr{EL}(S)$ of the attracting and repelling laminations for ψ.

Given $X \in \text{Teich}(S)$, and $\delta > 0$, we have from Corollary 1.4 that there is a neighborhood $U^+_\delta(\psi) \subset \mathscr{REL}(S)$ of λ^+ so that if $\lambda' \in U^+_\delta(\psi)$ is the support of the attracting fixed point $[\mu'] \in \mathscr{PML}(S)$ of another pseudo-Anosov element ψ', then there is a $Z \in A_\psi$ so that the ray $\mathbf{r}$ based at X with ending lamination λ' contains a segment within Hausdorff distance δ of a full period $g = \overline{Z\ \psi(Z)}$ of the action of ψ on A_ψ.

Thus we may argue by induction. Let $\{\psi_n\}_{n=1}^\infty \subset \text{Mod}(S)$ be a family of pseudo-Anosov elements whose corresponding closed geodesics on $\mathscr{M}(S)$ form a dense family in $\mathscr{M}^1(S)$, and let $X \in \text{Teich}(S)$ be a basepoint. Let $\delta_n \to 0$ be given so that the δ_n neighborhood of the axis A_n for ψ_n lies entirely within $\text{Teich}(S)$. It suffices to find a geodesic ray $\mathbf{r}$ based at X so that for each n there is a segment along $\mathbf{r}$ that lies within δ_n of the axis of some conjugate in $\text{Mod}(S)$ of ψ_n for a full period g_n along the axis.

Assume that for $k > 1$ we have a ray $\mathbf{r}_k$ based at X forward asymptotic to the axis of a conjugate $\hat{\psi}_k$ of ψ_k so that the support $\hat{\lambda}^+_k$ of the attracting lamination of $\hat{\psi}_k$ lies in the intersection

$$V_k = U_{\delta_1}(\hat{\psi}_1) \cap \ldots \cap U_{\delta_{k-1}}(\hat{\psi}_{k-1}).$$

Then for a sufficiently large power p_{k+1}, the support λ_{k+1} of the attracting lamination for ψ_{k+1} has image $\hat{\psi}_k{}^{p_{k+1}}(\lambda_{k+1})$ within V_k. Taking $\mathbf{r}_{k+1}$ to be the ray asymptotic to the axis of the pseudo-Anosov conjugate

$$\hat{\psi}_{k+1} = \hat{\psi}_k^{p_{k+1}} \circ \psi_{k+1} \circ \hat{\psi}_k^{-p_{k+1}}$$

of ψ_{k+1}, we have a ray asymptotic to the axis of a pseudo-Anosov element with attracting lamination in the intersection

$$V_{k+1} = U_{\delta_1}(\hat{\psi}_1) \cap \ldots \cap U_{\delta_k}(\hat{\psi}_k).$$

Thus, $\mathbf{r}_{k+1}$ lies within δ_n of the axis of the conjugate $\hat{\psi}_n$ of ψ_n, $n = 1, \ldots, k+1$, for a full period g_n along the axis of each. This completes the induction.

Thus any limit $\mathbf{r}_\infty$ of $\mathbf{r}_k$ as $k \to \infty$ in the visual sphere at X will have a dense trajectory in its projection $\mathscr{M}^1(S)$, provided, once again, that it is an infinite ray. But $\mathbf{r}_k$ passes within δ_n of the axis $\hat{A}_n$ of $\hat{\psi}_n$ at the segment $g_n \subset \hat{A}_n$, for each $k > n$, so the closest points $\mathbf{r}_k(t_n)$ to g_n range in a compact neighborhood in $\text{Teich}(S)$ of a bounded interval along $\hat{A}_n$. Thus, given $T > 0$, and n so that $t_n > T$, the segments $\mathbf{r}_k([0,T])$ sit as subsegments in a family of segments $\mathbf{r}_k(t_n)$ whose endpoints converge in $\text{Teich}(S)$ as $k \to \infty$. Thus, the sequence of geodesics $\mathbf{r}_k([0,T])$ converges to a geodesic in Teichmüller space for each T, by geodesic convexity of Teich(S) [Wol4]. It follows that the limit $\mathbf{r}_\infty$ is infinite and projects to a dense subset of $\mathscr{M}^1(S)$ as was claimed. $\square$

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