Asymptotics of Weil-Petersson geodesics II: bounded geometry and unbounded entropy

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June 28, 2010

Abstract

We use ending laminations for Weil-Petersson geodesics to establish that bounded geometry is equivalent to bounded combinatorics for Weil-Petersson geodesic segments, rays, and lines. Further, a more general notion of *non-annular bounded combinatorics*, which allows arbitrarily large Dehn-twisting, corresponds to an equivalent condition for Weil-Petersson geodesics. As an application, we show the Weil-Petersson geodesic flow has compact invariant subsets with arbitrarily large topological entropy.

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*Each author partially supported by the NSF. J. Brock was partially supported by a John Simon Guggenheim Foundation Fellowship.

1 Introduction

This paper is the second in a series analyzing the large-scale behavior of geodesics in the Weil-Petersson metric on Teichmüller space. In the first paper, [BMM], we defined a notion of an *ending lamination* for a Weil-Petersson geodesic ray, and gave a parametrization of the geodesic rays based at a fixed point $X \in \text{Teich}(S)$ that recur to the thick part of Teichmüller space in terms of their ending laminations as points in the Gromov-boundary of the curve complex. Our main goal in the present discussion is to describe a connection between ending laminations of *bounded type*, and the control they give over the trajectories of the rays to which they are associated.

Some of this is in direct analogy with *Teichmüller* geodesic rays. For these rays the notion of an ending lamination is explicit in the definition, and many of the questions we ask already have well-understood answers. The lack of a good description of the behavior of the hyperbolic structure of surfaces that lie along a Weil-Petersson geodesic has kept a full understanding of the large scale behavior of geodesics out of reach.

The main result of this paper is the equivalence of *bounded geometry* for a Weil-Petersson geodesic, which is just precompactness of its projection to the moduli space, and *bounded combinatorics* of its ending laminations, a notion analogous to bounded continued fraction expansions for slopes of torus foliations.

Specifically, given a bi-infinite Weil-Petersson geodesic, we consider ending laminations λ^+ and λ^- associated to its forward and backward trajectories. To each essential subsurface $Y \subsetneq S$ that is not a three-holed sphere, there is an associated coefficient

$$d_Y(\lambda^-, \lambda^+)$$

that gives a notion of distance between the projections of the ending laminations λ^+ and λ^- in the curve complex $\mathscr{C}(Y)$. We say the pair (λ^+, λ^-) has *K-bounded combinatorics* if there is an upper bound $K > 0$ to all such coefficients.

Theorem 1.1. (BOUNDED COMBINATORICS GEOMETRICALLY THICK) *For each $K > 0$ there is an $\varepsilon > 0$ so that if the ending laminations of a bi-infinite Weil-Petersson geodesic $\mathbf{g}$ have K-bounded combinatorics then $\mathbf{g}(t)$ lies in the ε-thick part for each t.*

We make precise the notion of bounded combinatorics in section 2 and remark that similar statements hold for geodesic segments and geodesic rays (see Theorem 4.1).

Conversely, constraining the geometry of surfaces along a Weil-Petersson geodesic forces a bound to the combinatorics of the ending laminations.

Theorem 1.2. (THICK GEODESICS COMBINATORIALLY BOUNDED) *Given $\varepsilon > 0$ there is a $K > 0$ so that if $\mathbf{g}$ is any bi-infinite geodesic in the ε thick part of* $\mathrm{Teich}(S)$*, then the combinatorics of the ending laminations associated to its ends are K-bounded.*

As part of the analysis we also have the following fellow travelling result for Teichmüller geodesics.

Theorem 1.3. *For all $\varepsilon > 0$ there is a $D > 0$ so that each bi-infinite ε-thick Weil-Petersson geodesic $\mathbf{g}$ lies at Hausdorff-distance D, in the Teichmüller metric, from a unique Teichmüller geodesic $\mathbf{h}$.*

See Theorem 3.2 for a more precise formulation.

Remark: After the announcement of the main results of this paper in the fall of 2007, Hamenstädt gave an elegant alternative proof of Theorems 1.1 and 1.2 [Ham3] via Teichmüller geodesics.

The case of non-annular bounds. As an intermediate step in the proof of Theorem 1.1 we start with the weaker assumption of *non-annular bounded combinatorics*, a criterion considered, for example, by Mahan Mj (see [Mj]) in the context of Kleinian groups, where the coefficients $d_Y(\lambda^+, \lambda^-)$ are bounded only for essential subsurfaces $Y \subsetneq S$ that are not annuli. This assumption allows for the possibility of arbitrarily large *relative twisting* of the ending laminations λ^+ and λ^- around various closed curves.

With this weaker assumption, we obtain a stability theorem for quasi-geodesics in the *pants graph*, a combinatorial model for the Weil-Petersson metric ([Br]). The pants graph $P(S)$, introduced by Hatcher and Thurston, encodes isotopy classes of pants decompositions of the surface S as its vertices, with edges joining vertices whose corresponding pants decompositions are related by certain elementary moves. By Theorem 1.1 of [Br], there is a quasi-isometry

$$Q\colon \mathrm{Teich}(S) \to P(S)$$

that associates to each $X \in \mathrm{Teich}(S)$ a shortest *Bers pants decomposition* for X.

Theorem 4.4. (STABILITY WITHOUT ANNULI) *Let $F\colon [0,T] \to P(S)$ be a K-quasi-geodesic, and let $F(0) = Q_-$ and $F(T) = Q_+$ denote its endpoints. If Q_-*

and Q_+ satisfy the non-annular bounded combinatorics condition, then $F(t)$ fellow travels a hierarchy path in $P(S)$.

Hierarchy paths will be discussed in Section 2, and the precise statement of Theorem 4.4 appears in §4. Stability of quasi-geodesics, standard in the setting of δ-hyperbolic metric spaces, only holds generally in $P(S)$ for low-complexity cases when the dimension $\dim_{\mathbb{C}}(\text{Teich}(S))$ is 1 or 2, (see [BF]). By the main result of [BM], a relative version holds for $\dim_{\mathbb{C}}(\text{Teich}(S)) = 3$.

Stability is also natural question in the context of Weil-Petersson geometry, as a Weil-Petersson geodesic determines a quasi-geodesic in the pants graph via the quasi-isometry. The corresponding condition for Weil-Petersson geodesics is more difficult to formulate, but it allows the possibility for geodesics to approach boundary strata over very small intervals of time by twisting. We will not need a result of this type, but the phenomenon of large twisting along short intervals approaching boundary strata, suggests why a bound on the amount of twisting, a condition not part of non-annular bounded combinatorics, but part of the assumption of K-bounded combinatorics, will be necessary to prove Theorem 1.1.

Topological entropy. In [BMM] we showed how ending laminations can be employed to understand fundamental features of the topological dynamics of the Weil-Petersson geodesic flow on the unit tangent bundle to moduli space (see [BMM, Thms. 1.8 and 1.9]). In this paper, we show how the finer combinatorial features of the ending laminations described above provide for further understanding of the flow. In particular we show

Theorem 1.4. (TOPOLOGICAL ENTROPY) *There are compact flow-invariant subsets of $\mathscr{M}^1(S)$ of arbitrarily large topological entropy.*

Here, $\mathscr{M}^1(S) = T^1\text{Teich}(S)/\text{Mod}(S)$ represents the quotient of the unit tangent bundle to $\text{Teich}(S)$ by the action of the mapping class group $\text{Mod}(S)$. We note that due to the fact that the Weil-Petersson geodesic flow is well defined for all time only on the lifts of bi-infinite geodesics in $\mathscr{M}(S)$ to $\mathscr{M}^1(S)$, topological entropy is not well defined on the whole of $\mathscr{M}^1(S)$. Nevertheless, the topological entropy for compact invariant subsets of the Weil-Petersson geodesic flow sits in strong contrast to the topological entropy for compact invariant subsets of the Teichmüller geodesic flow, which approach a positive supremum equal to the real dimension of the Teichmüller space in question (see [Ham2]). Theorem 1.4 follows from the following unboundedness result for the growth rate

$$p_\varphi(\mathscr{K}) = \liminf_{L\to\infty} \frac{\log n_{\mathscr{K}}(L)}{L}$$

of the number $n_{\mathscr{K}}(L)$ of closed orbits for the geodesic flow of length at most L in the invariant subset $\mathscr{K}$.

Theorem 1.5. (COUNTING ORBITS) *Given any $N > 0$, there is a compact Weil-Petersson geodesic flow-invariant subset $\mathscr{K} \subset \mathscr{M}^1(S)$ for which the asymptotic growth rate $p_\varphi(\mathscr{K})$ for the number of closed orbits in $\mathscr{K}$ satisfies*

$$p_\varphi(\mathscr{K}) \geq N.$$

Plan of the paper. The paper makes considerable use of the technology of *hierarchies* of geodesics in the curve complex $\mathscr{C}(S)$ and in the curve complexes $\mathscr{C}(W)$ of subsurfaces $W \subset S$ developed in [MM2]. Here, we axiomatize the idea of a *resolution* of such a hierarchy into a notion of *hierarchy path* in Section 2, where we also introduce other terminology we will need. Section 3 establishes Theorem 1.2 by a compactness argument. The same compactness argument also shows that a Weil-Petersson geodesic in the thick part of $\text{Teich}(S)$ fellow travels a Teichmüller geodesic in the Teichmüller metric. Section 4 establishes Theorem 4.4, and uses this to show Theorem 1.1, after an analysis of the combinatorial behavior of bounded length geodesics using recent work of Wolpert (Theorem 4.6). Finally, in Section 5, we apply the results of Section 4 to establish that there are compact geodesic-flow-invariant subsets of arbitrarily large topological entropy.

Acknowledgements. The authors thank the Mathematical Sciences Research Institute for their hospitality while much of the work in this paper was completed. The first author thanks Yale University for its hospitality and the John Simon Guggenheim Foundation for its generous support. We thank Ursula Hamenstädt and Scott Wolpert for illuminating conversations.

2 Preliminaries

In this section we review terminology and background, setting notation we will use.

Teichmüller space and its metrics. If S is a compact surface of negative Euler characteristic, the *Teichmüller space* $\text{Teich}(S)$ denotes the space of finite-area hyperbolic structures on $\text{int}(S)$ up to isotopy. By default we consider the Weil-Petersson metric on $\text{Teich}(S)$, which is defined via an L^2-norm on cotangent spaces given by

$$\|\varphi\|^2_{\text{WP}} = \int_X \frac{|\varphi|^2}{\rho_X}$$

where $\varphi \in T_X^*\mathrm{Teich}(S)$ is a holomorphic quadratic differential on X and ρ_X denotes the Poincaré metric on X. The induced Riemannian metric g_{WP} has been much studied by many authors, and we focus here on properties of its synthetic and geometry and distance function $d_{\mathrm{WP}}(.,.)$.

We will occasionally refer to the Teichmüller metric, a Finsler metric arising from an L^1 norm on cotangent spaces. Its distance function d_T measures the infimum over all quasiconformal maps in the natural isotopy class of the quasiconformal dilatation.

Pants decompositions and markings. A *pants decomposition* P in a surface S of finite type is a maximal collection of isotopy classes of disjoint, homotopically distinct, homotopically nontrivial, non-peripheral simple closed curves. A *marking* μ consists of a pants decomposition P which is the *base* of the marking, written base(μ), together with a collection of isotopy classes of *transversals* one for each curve in base(μ). For each $\alpha \in P$ the transversal curve α' intersects no curve in $P \setminus \alpha$, and intersects α a minimal number of times subject to this condition (i.e. once or twice depending on the topological type of $S \setminus (P \setminus \alpha)$). We will frequently blur the distinction between curves and their isotopy classes, as there is a unique geodesic representative in each isotopy class.

The Bers constant. Given S of negative Euler characteristic, we denote by $L_S > 0$ the constant so that for each $X \in \mathrm{Teich}(S)$, there is a *Bers pants decomposition* P_X of X determined by closed geodesics on X whose lengths are bounded by L_S. The isotopy classes of closed geodesics determining P_X are called *Bers curves* for X (see [Bus]). For each pants curve choose a minimal length transversal. The resulting marking is called a *Bers marking* and is denoted by ν_X. By the collar lemma [Bus], there are a bounded number of Bers pants decompositions and Bers markings on a given X. We call a curve that arises in a Bers pants decomposition P_X for X a *Bers curve* for X.

The complex of curves. The complex of curves $\mathscr{C}(S)$ serves to organize the isotopy classes of essential non-peripheral simple closed curves on S. Each is associated to a vertex of $\mathscr{C}(S)$, and k-simplices are associated to families of $k+1$ distinct isotopy classes that can be realized pairwise disjointly on S (there is an exception for 1-holed tori and 4-holed spheres, where 1-simplices correspond to pairs of vertices realized with intersection number 1 and 2, respectively). We make $\mathscr{C}(S)$ into a metric space by making each simplex Euclidean with sidelength 1, and letting $d_{\mathscr{C}}$ be the induced path metric. By the main result of [MM1], $(\mathscr{C}(S), d_{\mathscr{C}})$ is δ-hyperbolic, in the sense of Gromov.

A δ-hyperbolic space carries a natural *Gromov boundary*. In our setting of a path-metric space, points in this boundary are asymptote classes of quasi-geodesic rays, where two rays are asymptotic if their Hausdorff distance is finite. Klarreich showed [Kla] (see also [Ham1]) that the Gromov boundary of $\mathscr{C}(S)$ is identified with the space $\mathscr{E}\mathscr{L}(S)$ of *ending laminations* on S. We define $\mathscr{E}\mathscr{L}(S)$ by starting with Thurston's measured lamination space $\mathscr{M}\mathscr{L}(S)$, considering the subset of those laminations that *fill the surface* (namely, laminations $\mu \in \mathscr{M}\mathscr{L}(S)$ so that every essential simple closed curve γ satisfies $i(\mu,\gamma) > 0$) and forgetting the transverse measure on these laminations. The resulting quotient space is $\mathscr{E}\mathscr{L}(S)$, with the quotient, or "measure-forgetting," topology. Convergence from within $\mathscr{C}(S)$ to $\mathscr{E}\mathscr{L}(S)$ is also defined using this topology, considering $\mathscr{C}(S)$ as a subset of $\mathscr{M}\mathscr{L}(S)$ modulo measures.

The mapping class group, $\mathrm{Mod}(S)$, of orientation preserving homeomorphisms modulo those isotopic to the identity, acts naturally on $\mathscr{C}(S)$ via its action on the essential simple closed curves on S. Given a simplex $\sigma \in \mathscr{C}(S)$, we denote by

$$\mathrm{tw}(\sigma) < \mathrm{Mod}(S)$$

the free abelian group generated by Dehn twists about the curves represented by the vertices of σ.

The pants graph and marking graph. Central to our discussion here will be the quasi-isometric model for the Weil-Petersson metric obtained from the graph of pants decompositions on surfaces. The isotopy class of a pants decomposition P of S corresponds to a vertex of $P(S)$, and two vertices corresponding to pants decompositions P and P' are joined by an edge if they differ by an *elementary move*, namely, if P' can be obtained from P by replacing one of the isotopy classes of simple closed curves represented in the pants decomposition P with another that intersects it minimally. This defines a distance $d_P(\cdot,\cdot)$ in the pants graph.

Then there is a coarsely defined projection map

$$Q : \mathrm{Teich}(S) \to P(S)$$

that associates to each $X \in \mathrm{Teich}(S)$ a Bers pants decomposition on X.

Theorem 2.1. ([Br, Thm. 1.1]) *The map Q is a quasi-isometry.*

The *marking graph* $\widetilde{M}(S)$ is the graph whose vertices are markings (as above) and whose edges correspond to elementary moves which correspond to twists of transversals around pants curves, and (roughly) interchange of pants curves and transversals (see [MM2]). We denote the path metric associated to this graph by $d_{\widetilde{M}(S)}$. The relevant property for us is that $\widetilde{M}(S)$ is connected, and is acted on isometrically and cocompactly by the mapping class group. Note that $\widetilde{M}(S)$ "fibres" over $P(S)$ by the map that forgets the transversals.

Weil-Petersson geodesics. A *Weil-Petersson geodesic* is denoted $\mathbf{g}: J \to \mathrm{Teich}(S)$ where $J \subset \mathbb{R}$ is an interval, and $\mathbf{g}$ is geodesic parametrized by arclength, with respect to the Weil-Petersson distance d_{WP}.

Let $\alpha = \inf J$ and $\omega = \sup J$. If $\omega \in J$ (respectively $\alpha \in J$) we say the forward (respectively backward) end of $\mathbf{g}$ is closed. If $\omega \notin J$ we require that $\mathbf{g}$ cannot be extended past ω (i.e. $\mathbf{g}(t)$ exits every compact set in $\mathrm{Teich}(S)$ as $t \to \omega$), and we say the forward end of $\mathbf{g}$ is open; and similarly for the backward end.

We call $\mathbf{g}$ a *segment* if $J = [\alpha, \omega]$, a *ray* if $J = [\alpha, \omega)$ or $J = (\alpha, \omega]$, and a *line* if $J = (\alpha, \omega)$. If J (hence $\mathbf{g}$) has infinite length, we call $\mathbf{g}$ an unbounded ray or line.

If the forward (resp. backward) end of $\mathbf{g}$ is closed we denote by $\nu^+(\mathbf{g})$ a choice of Bers marking $\nu_{\mathbf{g}(\omega)}$ for the surface $\mathbf{g}(\omega)$. (resp. $\nu^-(\mathbf{g}) = \nu_{\mathbf{g}(\alpha)}$). For open ends we have the notion of *ending lamination*, which we will define presently.

Geodesic length functions and ending laminations for rays. To each isotopy class of essential non-peripheral simple closed curves there is an associated *geodesic length function*

$$\ell_\alpha : \mathrm{Teich}(S) \to \mathbb{R}_+$$

that assigns to each $X \in \mathrm{Teich}(S)$ the arclength $\ell(\alpha^*)$ of the geodesic representative α^* of α on X. Given a path $\mathbf{g}(t)$ in $\mathrm{Teich}(S)$, ℓ_α determines a natural *length function* along a geodesic $\mathbf{g}$:

$$\ell_{\mathbf{g},\alpha}(t) = \ell_\alpha(\mathbf{g}(t)).$$

When $\mathbf{g}(t)$ is a Weil-Petersson geodesic, it is due to Wolpert (see [Wol2]) that the function $\ell_{\mathbf{g},\alpha}(t)$ is strictly convex.

In [BMM] we study the following definitions for a Weil-Petersson geodesic ray $\mathbf{r}$.

Definition 2.2. *An* ending measure *for a geodesic ray* $\mathbf{r}(t)$ *is any limit* $[\mu]$ *in* $\mathscr{PML}(S)$*, Thurston's space of projective measured laminations, of the projective classes* $[\gamma_n]$ *of any infinite family of distinct Bers curves for* $\mathbf{r}$.

We pay special attention to simple closed curves whose length decay to zero.

Definition 2.3. *A simple closed curve γ is a* pinching curve *for* $\mathbf{r}$ *if* $\ell_{\mathbf{r},\gamma}(t) \to 0$ *as* $t \to \omega$.

Taking the union of the support of all ending measures together with the pinching curves for $\mathbf{r}$ we obtain the *ending lamination* $\lambda(\mathbf{r})$. That this is in fact a lamination follows from [BMM, Prop. 2.9], which states:

Theorem 2.4. *If* $\mathbf{r}(t)$ *is a Weil-Petersson geodesic ray, the pinching curves and supports of ending measures for* $\mathbf{r}$ *have no transverse intersection. Hence their union* $\lambda(\mathbf{r})$ *is a geodesic lamination.*

Ending data. For each open end of a geodesic $\mathbf{g}$ we thus have an ending lamination; we denote these by $\lambda^+(\mathbf{g})$ for the forward (ω) end and $\lambda^-(\mathbf{g})$ for the backward (α) end.

If (say) the forward end of $\mathbf{g}$ is closed, so that $\mathbf{g}(\omega) \in \text{Teich}(S)$, then we let $\nu^+(\mathbf{g})$ denote a Bers marking for $\mathbf{g}(\omega)$ (if there are several we pick one arbitrarily). Define $\nu^-(\mathbf{g})$ similarly. In general we call $\nu^\pm(\mathbf{g})$ or $\lambda^\pm(\mathbf{g})$ the *ending data* of $\mathbf{g}$, and if we do not wish to be specific about whether they are markings or laminations we use the notation $\nu^\pm$.

The completion of the Weil-Petersson metric and its strata. The failure of completeness of the Weil-Petersson, due to Wolpert and Chu (see [Wol1] and [Chu]) arises from finite length paths in the Weil-Petersson metric that leave every compact set corresponding to "pinching deformations" where the length of a family of simple closed curves tends to zero. If the hyperbolic metrics on the complement of this family of curves converge to finite area hyperbolic structures on their complementary subsurfaces one may identify this *nodal surface* with the corresponding point in the Weil-Petersson completion. Masur showed that the completion is naturally the augmented Teichmüller space (see [Brs, Ab]) obtained by adjoining *boundary strata* corresponding to (products of) lower dimensional Teichmüller spaces.

These strata and their adjunction are best understood by extending geodesic length functions ℓ_α to allow their vanishing. Precisely, if P is a pants decomposition of S then P determines a maximal simplex in $\mathscr{C}(S)$, and a *frontier space subordinate to* P, $\mathscr{S}_\sigma$, is determined by the vanishing of the length functions $\{\ell_\alpha = 0\}$ for the simple closed curves α representing the vertices of σ. The topology on the union of $\text{Teich}(S)$ with the frontier spaces subordinate to P is given by *extended Fenchel-Nielsen coordinates* for $\text{Teich}(S) \cup \mathscr{S}_\sigma$ in which the length parameters ℓ_α, are extended to take values in $\mathbb{R}_{\geq 0}$, and twist parameters θ_α are undefined on

$\mathscr{S}_\sigma$ for each $\alpha \in \sigma^0$. Then the union is topologized by the requirement that ℓ_α vary continuously. See [BMM] [Wol3].

Hierarchy paths and the distance formula. Though the pants complex gives a coarse notion of distance in the Weil-Petersson metric (via Theorem 2.1), it is not at all clear what form distance minimizing paths may take. Nevertheless, a kind of combinatorial formula to estimate pants distance arises out of consideration of the curve complex $\mathscr{C}(S)$ and the curve complexes $\mathscr{C}(Y)$ of subsurfaces $Y \subset S$ considered simultaneously.

First of all, a notion of *projection to a subsurface* is defined: given a proper, essential subsurface $W \subset S$, there is a projection

$$\pi_W : \mathscr{C}(S) \to \mathscr{P}(\mathscr{C}(W))$$

from the curve complex $\mathscr{C}(S)$ to the power set of the curve complex $\mathscr{C}(W)$ as follows: for any $\gamma \in \mathscr{C}(S)$ with γ isotopic into the complement of W, we set $\pi_W(\gamma) = \emptyset$. Now assuming W is not an annulus: If γ is isotopic into W then we set $\pi_W(\gamma) = \gamma$. Otherwise, after isotoping γ to minimize the number of components of $\gamma \cap W$, we take for each arc a of the intersection the boundary components of a regular neighborhood of $a \cup \partial W$ which are essential curves in W. The union of these is $\pi_W(\gamma)$.

The case that W is an annulus is slightly different: here we let $\mathscr{C}(W)$ denote the complex whose vertices are arcs connecting the boundaries of W up to isotopy rel endpoints, and whose edges are pairs of arcs with disjoint interiors. If γ intersects W essentially we lift it to the annular cover associated to W, which we identify with W, and let $\pi_W(\gamma)$ be the union of components of the lift that connect the boundary components of W. Note that $\mathscr{C}(W)$ is quasi-isometric to $\mathbb{Z}$ and its distance function measures a coarse form of twisting around the annulus W. This is sometimes called the twist complex of W.

The *projection distances*

$$d_W(\sigma, \gamma) = \mathrm{diam}(\pi_W(\sigma), \pi_W(\gamma))$$

give a useful notion of the relative distance between simplices σ and γ as seen from the subsurface W. We can define this just as well when σ or γ are markings. Then one has the following Lipschitz property for the projections π_W (see [MM2, Lemma 2.3]):

Proposition 2.5. *For any simplex $\sigma \in \mathscr{C}(S)$ and any subsurface $W \subset S$, if $\pi_W(\sigma) \neq \emptyset$ then we have*

$$\mathrm{diam}_{\mathscr{C}(W)}(\pi_W(\sigma)) \leq 2.$$

Similarly, if μ and μ' are pants decompositions or markings on S differing by an elementary move, then we have

$$d_W(\mu,\mu') \leq 4.$$

There is a strong relationship between the geometry of geodesics in such curve complexes, projection distances, and a certain type of efficient path in $P(S)$ called a *hierarchy path*. These considerations, developed in [MM2] can be summarized in the following theorem.

Given $n \geq 0$ let $[[n]]_M$ denote the quantity

$$[[n]]_M = \begin{cases} n & \text{if } n \geq M, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Further, given $c_1 > 1$ and $c_2 > 0$ we denote by $\asymp_{c_1,c_2}$ equality up to multiplicative error c_1 and additive error c_2. In other words, we write $x \asymp_{c_1,c_2} y$ whenever

$$\frac{x}{c_1} - c_2 \leq y \leq c_1 x + c_2.$$

Theorem 2.6. (HIERARCHY PATHS) *Given pants decompositions P_1 and P_2 in $P(S)$, there is a path $\rho\colon [0,n] \to P(S)$ joining $\rho(0) = P_1$ to $\rho(n) = P_2$ with the following properties.*

1. *There is a collection $\{Y\}$ of essential, non-annular subsurfaces of S, called* component domains *for ρ, so that for each component domain Y there is a connected interval $J_Y \subset [0,n]$ with $\partial Y \subset \rho(j)$ for each $j \in J_Y$.*

2. *There is an $M_1 > 0$ so that for each essential subsurface $Y \subset S$ with $d_Y(P_1,P_2) > M_1$, Y is a component domain for ρ.*

3. *For component domain Y, there is a geodesic $g_Y \subset \mathscr{C}(Y)$ so that for each $j \in J_Y$, there is an $\alpha \in g_Y$ with $\alpha \in \rho(j)$.*

4. *There is an $M_2 > 0$ so that if $J_Y = [t_1,t_2]$ and $t > t_2$ then $d_Y(\rho(t),\rho(t_2)) < M_2$ while for $t < t_1$ we have $d_Y(\rho(t),\rho(t_1)) < M_2$.*

5. (THE DISTANCE FORMULA) *Given any $M_3 \geq M_1$, there exists $c_1 > 1$ and $c_2 > 0$, so that*

$$d_P(P_1,P_2) \asymp_{c_1,c_2} \sum\nolimits_Y' [[d_Y(P_1,P_2)]]_{M_3}.$$

where the notation $\sum_Y'$ indicates that the sum is taken over all non-annular *subsurfaces Y including S itself.*

6. *There is a $K_H > 1$ so that the path ρ satisfies*

$$\frac{1}{K_H} \leq \frac{d_P(\rho(i),\rho(j))}{|j-i|} \leq 1. \tag{2.1}$$

The whole surface S is always a component domain for ρ. The geodesic g_S is called the main geodesic *for the hierarchy path ρ.*

Given P_1 and P_2 in $P(S)$, we denote by $\rho = \rho(P_1, P_2)$ an arbitrary choice of hierarchy path joining P_1 to P_2.

These hierarchy paths in $P(S)$ are *resolution sequences*, so called, of the *hierarchies without annuli* defined in [MM2, §8]. The main construction of [MM2] actually takes place in the marking graph $\widetilde{M}(S)$, and Theorem 2.6 can be restated for hierarchy paths $\rho\colon [0,n] \to \widetilde{M}(S)$ in the marking graph, where surfaces Y are allowed to be annuli. In particular, there is a distance formula for marking distance $d_{\widetilde{M}}$ which takes the same form as (5), but the sum is taken over all essential subsurfaces (see [MM2, Thm. 6.12]).

There will be instances in the paper where it is appropriate to consider hierarchy paths in ρ in $P(S)$ and in $\widetilde{M}(S)$, and we will make it clear from context which is being considered. The main construction of [MM2] also allows for the case when the main geodesic g_S is infinite or bi-infinite, in which case the geodesic, and hence the hierarchy path, describes curve systems that are asymptotic to a lamination or pair of laminations in $\mathscr{EL}(S)$ in the measure forgetting topology.

Bounded combinatorics. Let σ and σ' be simplicies in $\mathscr{C}(S)$ or let them be markings. We say σ and σ' have *K-bounded combinatorics* if for each essential subsurface $W \subsetneq S$ which is intersected by both σ and σ', the projection distance satisfies

$$d_W(\sigma,\sigma') < K.$$

We note that for any hierarchy path $\rho\colon [0,n] \to P(S)$ or hierarchy path $\rho\colon [0,n] \to \widetilde{M}(S)$ whose endpoints satisfy

$$d_W(\rho(0),\rho(n)) < K$$

an application of the triangle inequality together with part (4) of Theorem 2.6 guarantees that for each i, j in $[0,n]$, we have

$$d_W(\rho(i),\rho(j)) < K + 2M_2. \tag{2.2}$$

Recurrence for rays Given $X \in \text{Teich}(S)$ we let $\text{sys}(X)$ denote the length of the shortest closed geodesic in X. A Weil-Petersson geodesic ray $\mathbf{r}$ is said to be *recurrent* if there is an $\varepsilon > 0$ and a family of times $t_n \to \infty$ for which the hyperbolic surface $\text{sys}(\mathbf{r}(t_n)) > \varepsilon$. It was shown in [BMM] that the ending lamination uniquely determines the asymptote class of for a recurrent ray. Precisely, we have

Theorem 2.7. *Let* $\mathbf{r}$ *and* $\mathbf{r}'$ *be two geodesic rays with* $\lambda(\mathbf{r}) = \lambda(\mathbf{r}')$. *If* $\mathbf{r}$ *is recurrent, then* $\mathbf{r}$ *and* $\mathbf{r}'$ *are strongly asymptotic.*

Here, strongly asymptotic refers to the existence of parametrizations $\mathbf{r}(s)$ and $\mathbf{r}'(s)$ (not necessarily by arclength) for which $d(\mathbf{r}(s), \mathbf{r}'(s)) \to 0$.

Given $\varepsilon > 0$, a geodesic segment, ray, or line $\mathbf{g}$ is said to have ε-bounded geometry if the length of the shortest closed geodesic on $\mathbf{g}(t)$ is bounded below by ε for each t for which $\mathbf{g}(t)$ is defined. Then we observe that as a direct consequence of Theorem 2.7 we have the following.

Corollary 2.8. *Let* $\mathbf{r}$ *be a geodesic ray with* ε*-bounded geometry. Then if* $\mathbf{r}'$ *satisfies* $\lambda(\mathbf{r}) = \lambda(\mathbf{r}')$ *then* $\mathbf{r}$ *and* $\mathbf{r}'$ *are strongly asymptotic.*

The following Proposition combining Lemma 2.10 and Corollary 2.12 of [BMM] will be useful for our purposes.

Proposition 2.9. *Let* $\mathbf{r}_n \to \mathbf{r}_\infty$ *be a sequence of segments or rays based at a fixed* $X \in \text{Teich}(S)$ *that converge in the visual sphere, and assume* $\mathbf{r}_\infty$ *has an ending measure* μ. *If* $\mathbf{r}_n$ *is a segment, let* μ_n *be a Bers pants decomposition for its endpoint. If* $\mathbf{r}_n$ *is a ray, let* μ_n *be any ending measure or pinching curves for* $\mathbf{r}_n$. *Let* $\mu' \in \mathscr{ML}(S)$ *a representative of any limit* $[\mu']$ *of projective classes* $[\mu_n]$ *in* $\mathscr{PML}(S)$. *Then we have*

$$i(\mu, \mu') = 0.$$

In particular, if μ *fills the surface, then* $|\mu| = |\mu'|$.

We note that Lemma 2.9 and Lemma 2.10 in [BMM] are not stated for segments. However the proofs are verbatim true if we allow $\mathbf{r}_n$ to be a segment.

3 Bounded geometry implies bounded combinatorics

Let $\mathscr{K}$ denote a compact subset of the moduli space $\mathscr{M}(S)$. If $\mathbf{g}$ is a Weil-Petersson geodesic segment, ray or line whose projection to $\mathscr{M}$ lies in $\mathscr{K}$, then we say $\mathbf{g}$ is *cobounded* or $\mathscr{K}$*-cobounded*. Let $\nu^\pm = \nu^\pm(\mathbf{g})$ denote the ending data of $\mathbf{g}$ (markings or laminations, as in Section 2).

Theorem 3.1. *If* $\mathbf{g}$ *is* $\mathscr{K}$*-cobounded then there is a* K *depending only on* $\mathscr{K}$ *so that the ending data* $\nu^{\pm}$ *of* $\mathbf{g}$ *satisfy the bounded combinatorics condition:*

$$d_W(\nu^+, \nu^-) \leq K$$

where W *is any essential proper subsurface of* S*.*

We will also deduce the following.

Theorem 3.2. *For all* $\varepsilon > 0$ *there is* $D > 0$ *so that each bi-infinite* ε *thick Weil-Petersson geodesic* $\mathbf{g}$ *lies at Hausdorff distance* D *in the Teichmüller metric from a unique Teichmüller geodesic* $\mathbf{h}$*.*

Proof of Theorem 3.1. Let $\mathbf{G}_{\mathscr{K}}$ be the set of $\mathscr{K}$-cobounded Weil-Petersson geodesics which contain 0 in their parameter interval. Note that each end of such a geodesic is either infinitely long, with ending lamination λ^+ or λ^-, or closed, terminating in a point of Teich(S) with Bers marking ν^+ or ν^- (that is, there are no pinching curves). In the latter case we may consider any of the (finitely many) Bers pants decompositions of the endpoint.

We will now follow a compactness argument of Mosher [Msh] to establish a bound on the combinatorics asscoiated to its Bers markings or ending laminations.

Consider the subset

$$\Gamma \subset \mathbf{G}_{\mathscr{K}} \times \mathscr{ML} \times \mathscr{ML}$$

consisting of triples $(\mathbf{g}, \mu^+, \mu^-)$ such that

- μ^+ is a measure on the lamination λ^+ if the forward end $\mathbf{g}_+$ is infinite, and on a Bers pants decomposition if it is finite; and similarly for μ^- and $\mathbf{g}_-$.
- $\mu^{\pm}$ have length 1 with respect to the hyperbolic structure on $\mathbf{g}(0)$.

We let Γ inherit the product topology, where we put Thurston's topology on $\mathscr{ML}$ and give $\mathbf{G}_{\mathscr{K}}$ the topology of convergence of parameter intervals together with uniform convergence on compact subsets.

We first show that the action of Mod(S) on Γ is co-compact. Let $\mathscr{K}_0$ be a compact fundamental domain for the action of Mod(S) on the preimage of $\mathscr{K}$ in Teich(S), and let $\mathbf{G}_{\mathscr{K}_0}$ be the set of $\mathbf{g} \in \mathbf{G}_{\mathscr{K}}$ with $\mathbf{g}(0) \in \mathscr{K}_0$. Clearly $\mathbf{G}_{\mathscr{K}_0}$ is compact, and every point in Γ can be moved by Mod(S) into $\Gamma_0 = \Gamma \cap \mathbf{G}_{\mathscr{K}_0} \times \mathscr{ML} \times \mathscr{ML}$.

Let $(\mathbf{g}_n, \mu_n^+, \mu_n^-) \in \Gamma_0$ be a sequence such that $\mathbf{g}_n \to \mathbf{g} \in \mathbf{G}_{\mathscr{K}_0}$. Since $\mu_n^\pm$ have length 1 at $\mathbf{g}_n(0)$ and $\mathbf{g}_n(0) \to \mathbf{g}(0)$, we may conclude that, after restricting to a subsequence, $\mu_n^\pm$ converge in $\mathscr{ML}$ to $\mu^\pm$ with length 1 at $\mathbf{g}(0)$.

If $\mathbf{g}_+$ is finite then μ^+ is a measure on a Bers pants decomposition for the endpoint, and similarly for the backward end $\mathbf{g}_-$.

If $\mathbf{g}_+$ is infinite we must show that μ^+ is a measure on the ending lamination λ^+. We claim that the length of μ_n^+ is uniformly bounded on $\mathbf{g}_n(t)$ for $t \geq 0$. If $\mathbf{g}_n$ is finite in the forward direction this is a consequence of convexity of the length function. If not, then since $\mathbf{g}_n$ is recurrent (being cobounded), we can apply Lemma 4.5 of [BMM], ensuring that a measured lamination has bounded length along a recurrent ray if and only if its support is the ending lamination. Hence μ_n^+ is bounded along $\mathbf{g}_n$ in the forward direction, and by convexity it is bounded by 1 for $t \geq 0$. It follows in the limit that μ^+ has bounded length along $\mathbf{g}_+$, and hence (again by Lemma 4.5 of [BMM]) its support is its ending lamination. The same applies to μ^- and $\mathbf{g}_-$, and so we conclude that $(\mathbf{g}, \mu^+, \mu^-) \in \Gamma_0$. This proves that Γ is co-compact.

Now consider the set of quadruples $(\mathbf{g}, \mu^+, \mu^-, W)$ where $(\mathbf{g}, \mu^+, \mu^-) \in \Gamma$ and W is a proper essential subsurface. Let us first consider the case of bi-infinite geodesics: The length $\ell_{\mathbf{g}, \partial W}(t)$ is a proper convex function of t and hence has a unique minimum. After reparameterizing and rescaling the $\mu^\pm$ we may assume that ∂W has minimum length at $\mathbf{g}(0)$.

Suppose that our desired bound on $d_W(\mu^+, \mu^-)$ fails and there is a sequence $(\mathbf{g}_n, \mu_n^+, \mu_n^-, W_n)$, normalized in this way, such that $d_{W_n}(\mu_n^+, \mu_n^-) \to \infty$. We will find a contradiction. The cocompactness of Γ tells us that, after acting by $\mathrm{Mod}(S)$ and restricting to a subsequence, we can assume that $(\mathbf{g}_n, \mu_n^+, \mu_n^-) \to (\mathbf{g}, \mu^+, \mu^-)$ in Γ, which must still be bi-infinite. We may also assume that $\{\partial W_n\}$ converges in $\mathscr{PML}(S)$, to a projectivized measured lamination represented by $\sigma \in \mathscr{ML}(S)$. Continuity of length on $\mathrm{Teich}(S) \times \mathscr{ML}(S)$ and convexity in the limit, implies that $\ell_{\mathbf{g}, \sigma}(t)$ still has a minimum at $t = 0$. Hence σ cannot have support equal to either μ^+ or μ^-, since Lemma 4.5 of [BMM] ensures that a measured lamination can only be supported on the ending lamination of a recurrent ray if its length goes to 0 along the ray.

Since μ^+ and μ^- are filling and minimal, it follows that they intersect σ transversely, and that σ cuts the leaves of $\mu^\pm$ into segments whose lengths admit some upper bound. A limit of laminations in $\mathscr{ML}(S)$ always has support contained in any Hausdorff limit of supports of its approximates.

Now first assume that W_n are *not* annuli. It follows that in the sequence $\mu_n^\pm$ are cut up by ∂W_n into pieces of bounded length. Therefore any two of these pieces intersect a bounded number of times (usually 0), and this bounds $d_{W_n}(\mu_n^+, \mu_n^-)$, a contradiction.

Now assume that W_n are annuli. Since μ^+, μ^- intersect σ transversely, their approximates $\mu_n^\pm$ make a definite angle with the approximates W_n of σ. There is a lower bound on the length of the geodesic representing W_n. It follows that any lift of a leaf of μ_n^+ to the annular cover corresponding to a component of W_n has intersection bounded above with any other leaf that crosses W_n. This again gives a contradiction to the assumption that the projections go to infinity.

When $\mathbf{g}$ has endpoints, the minimum of $\ell_{\mathbf{g},\partial W}$ can occur at the endpoint, and the same can occur in the limit. However since the geodesics are co-compact, the minimum is bounded away from 0. The same argument still shows that, in the limit, σ cannot be supported on an ending lamination of an infinite end. A marking intersects *every* lamination (the components of the marking base are intersected by the transversals), so the same contradiction can be obtained. $\square$

Proof of Theorem 3.2. By [GM], for any pair (F_h, F_v) of measured laminations that bind S there is a unique surface $X = X(F_h, F_v) \in \mathrm{Teich}(S)$ and quadratic differential $q = q(F_h, F_v)$, holomorphic on X, whose horizontal and vertical measured foliations are equivalent to F_h and F_v respectively (via the usual equivalence between measured foliations and laminations). The family $X(t) = X(e^t F_h, e^{-t} F_v)$ is a Teichmüller geodesic parameterized by arclength (and all Teichmüller geodesics are obtained this way). Note actually that $X(kF_h, kF_v) = X(F_h, F_v)$ for any $k > 0$, since the two constructions differ only by a conformal factor. Hence *any* two multiples of F_h and F_v yield points on the same Teichmüller geodesic.

For each $(\mathbf{g}, \mu^+, \mu^-) \in \Gamma$, the laminations μ^+ and μ^- bind the surface (by Corollary 4.6 of [BMM]) so we can therefore associate the (parameterized) Teichmüller geodesic

$$\mathbf{h}(t) = X(e^t \mu^+, e^{-t} \mu^-).$$

Now we wish to prove that $d_T(\mathbf{g}(t), \mathbf{h})$ is bounded (uniformly on Γ). The map that assigns to each $(\mathbf{g}, \mu^+, \mu^-) \in \Gamma$ the point $(\mathbf{g}(0), \mathbf{h}(0)) \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ is $\mathrm{Mod}(S)$-equivariant and continuous on the co-compact set Γ. Thus for some M we have

$$d_T(\mathbf{g}(0), \mathbf{h}(0)) \leq M$$

for all points in Γ. Now let t be any parameter value in the domain of $\mathbf{g}$ and define the geodesic $\mathbf{g}_t$ by $\mathbf{g}_t(s) = \mathbf{g}(s+t)$, so that $\mathbf{g}_t(0) = \mathbf{g}(t)$. Let μ_t^+, μ_t^- be the multiples of μ^+, μ^- that have length 1 on $\mathbf{g}_t(0)$; then we have $(\mathbf{g}_t, \mu_t^+, \mu_t^-) \in \Gamma$. The corresponding Teichmüller geodesic $\mathbf{h}_t$ satisfies $\mathbf{h}_t(0) = \mathbf{h}(s)$ for some s.

Then the above says that we have

$$d_T(\mathbf{g}(t),\mathbf{h}(s)) = d_T(\mathbf{g}_t(0),\mathbf{h}_t(0)) \le M.$$

This shows that $\mathbf{g}$ lies in an M-neighborhood of $\mathbf{h}$, for all $(\mathbf{g},\mu^+,\mu^-) \in \Gamma$. It remains to obtain a bound in the other direction. Given $\mathbf{g}$ with parameter interval J, for each integer point $n \in J \cap \mathbb{Z}$ let s_n be a point in the domain of $\mathbf{h}$ such that $d_T(\mathbf{g}(n),\mathbf{h}(s_n)) \le M$. Since $\mathbf{g}$ is $\mathscr{K}$-cobounded, the *Teichmüller* distance $d_T(\mathbf{g}(n),\mathbf{g}(n+1))$ is bounded by some M'. Hence there is a uniform upper bound on $d_T(\mathbf{h}(s_n),\mathbf{h}(s_{n+1}))$, and so $\mathbf{h}([s_n,s_{n+1}])$ lies in a uniform neighborhood of $\mathbf{g}$, guaranteeing that $\mathbf{h}$ lies in a uniform neighborhood of $\mathbf{g}$ in the Teichmüller metric. □

4 Bounded combinatorics implies bounded geometry

In this section we will prove the converse to Corollary 3.1, namely that a Weil-Petersson geodesic segment, ray, or line whose end-invariants have bounded combinatorics must have bounded geometry:

Theorem 4.1. *Given $K > 0$ and a compact $\mathscr{K}_0 \subset \mathscr{M}(S)$, there is a compact $\mathscr{K} \subset \mathscr{M}(S)$ such that the following holds: Let $\mathbf{g}$ be a geodesic segment ray or line with finite endpoints, if any, projecting to $\mathscr{K}_0$ and ending data $\nu^\pm$. If*

$$d_W(\nu^+,\nu^-) \le K \tag{4.1}$$

for all proper essential subsurfaces W, then $\mathbf{g}$ is $\mathscr{K}$-cobounded.

We will prove this first in the case that $\mathbf{g}$ is a finite segment, and in §4.4 generalize to rays and lines. The first step, in §4.1, is to use a "stability of quasi-geodesics" argument to argue that the geodesic $\mathbf{g}$ must remain within a bounded Weil-Petersson distance of a path arising from a hierarchy path in $P(S)$ connecting its endpoints. Indeed this will hold not just with the general bound on projections but with the weaker assumption of that only the non-annular projections are bounded. Theorem 4.4 will give the combinatorial version of this stability statement.

In §4.2-4.3 we deduce the full strength of Theorem 4.1, which in view of Theorem 4.4 corresponds to showing that the geodesic stays away from the strata of the completion that are combinatorially close to the hierarchy path. A result of Wolpert (Theorem 4.6) will be used in §4.3 to show that under these circumstances, close approaches to these strata force the buildup of Dehn twists in certain curves, which (together with the information from Theorem 4.4) will contradict the bound on annular projections.

4.1 Projections to hierarchies and stability

If Q and $Q' \in P(S)$ are pants decompositions, we let $\rho = \rho(Q, Q')$ denote a hierarchy path $\rho\colon [0,n] \to P(S)$, as in Theorem 2.6, with $\rho(0) = Q$ and $\rho(n) = Q'$. The choice of ρ is not unique, but we will be satisfied with making an arbitrary one.

Let $|\rho| \subset P(S)$ denote the union

$$|\rho| = \cup_{i=0}^{n} \rho(i),$$

namely the image of the hierarchy path in $P(S)$.

We will at times consider pants decompositions P as maximal simplices in $\mathscr{C}(S)$. In particular, as with proper subsurfaces, we will employ the notation

$$d_S(P, P') = \operatorname{diam}(\pi_S(P) \cup \pi_S(P'))$$

where

$$\pi_S\colon P(S) \to \mathscr{C}(S)$$

denotes the projection of $P(S)$ into $\mathscr{C}(S)$ that associates to a pants decomposition P the maximal simplex in $\mathscr{C}(S)$ determined by its simple closed curves, and the diameter is taken in the metric on $\mathscr{C}(S)$.

The following Lemma shows that under the bounded combinatorics assumption, the mapping $\pi_S \circ \rho$ determines a quasi-geodesic in $\mathscr{C}(S)$.

Lemma 4.2. *Given $K > 0$, let ρ be a hierarchy path satisfying the non-annular K-bounded combinatorics condition, namely*

$$d_W(\rho(0), \rho(n)) \leq K. \tag{4.2}$$

for all proper non-annular essential subsurfaces $W \subset S$. Then there is a $c_1' > 1$ so that

$$\frac{1}{c_1'} \leq \frac{d_S(\rho(i), \rho(j))}{d_P(\rho(i), \rho(j)} \leq c_1' \tag{4.3}$$

for $i \neq j$.

Proof. Recall the consequence (2.2) of Definition 2.6, that we have for any i and j the bound

$$d_W(\rho(i),\rho(j)) \leq K + 2M_2$$

for any non-annular proper subsurface W.

Taking $M_3 > K + M_1 + 2M_2 + 2$, the distance formula (Theorem 2.6, part (5)) guarantees there are c_1 and c_2 depending on M_3 so that we have the estimate

$$d_P(\rho(i),\rho(j)) \asymp_{c_1,c_2} \sum\nolimits'_V [[d_V(\rho(i),\rho(j))]]_{M_3}, \tag{4.4}$$

where we recall the sum is over non-annular subsurfaces. But all the terms in the sum other than the $V = S$ term are beneath the threshold M_3, so we have

$$d_P(\rho(i),\rho(j)) \asymp_{c_1,c_2} d_S(\rho(i),\rho(j)). \tag{4.5}$$

Since $\rho(i)$ and $\rho(j)$ are pants decompositions, the term $d_S(\rho(i),\rho(j))$ is always positive. By Theorem 2.6, part (6), $d_P(\rho(i),\rho(j))$ is always positive when $i \neq j$ so in fact there is a c'_1 so that (4.3) holds.

□

We now define a "projection"

$$\pi_\rho : P(S) \to |\rho| \tag{4.6}$$

as follows: Given P in $P(S)$ let β be any choice of vertex of P. Let $v = \pi_m(\beta)$ be any closest point to β in m, with respect to the metric of $\mathscr{C}(S)$, where m is the main geodesic of the hierarchy path ρ, and then let $\pi_\rho(P)$ be any choice of $\rho(i)$ for which $\rho(i)$ contains v. The δ-hyperbolicity of $\mathscr{C}(S)$ implies that v is well-defined up to uniformly bounded ambiguity, and if ρ satisfies the non-annular bound (4.2) then (4.3) implies that $\rho(i)$ is defined up to bounded ambiguity as well.

When ρ satisfies (4.2) we will prove that π_ρ is "coarsely contracting" in the following sense:

Theorem 4.3. *Given K there exist N, R_0, and $C > 0$ such that if $\rho = \rho(Q_+, Q_-)$ is a hierarchy path satisfying the non-annular bounded combinatorics property (4.2), then the projection π_ρ satisfies*

1. *For $P \in |\rho|$, $d_P(P, \pi_\rho(P)) \leq N$*
2. *If $d_P(P_0, P_1) \leq 1$, then $d_P(\pi_\rho(P_0), \pi_\rho(P_1)) \leq N$*

3. *If* $d_P(P,|\rho|) = R \geq R_0$*, then*

$$\operatorname{diam}\left(\pi_\rho(\mathcal{N}_{R/C}(P))\right) \leq N.$$

Here distances and diameters are all taken in $P(S)$*, and* $\mathcal{N}_r$ *denotes a neighborhood of radius* r *in* $P(S)$*.*

Remark: the result also holds in the full marking graph if we require the bound (4.2) for annular surfaces as well.

Proof. For Conclusion (1), we note that for $P \in |\rho|$, the diameter of the closest point set to P on m has diameter at most 1. Then $d_P(P,\pi_\rho(P)) \leq c_1'$, by the lower bound in (4.3).

Conclusion (2) follows from δ-hyperbolicity of $\mathscr{C}(S)$, together with an application of (4.3). In particular, it is a standard property of δ-hyperbolic spaces that there is an L_δ depending only on the hyperbolicity constant δ so that the nearest point projection to a geodesic is L_δ-Lipschitz. If $d_P(P_0,P_1) \leq 1$, then for any $\alpha \in P_0$ and $\beta \in P_1$ we have

$$d_S(\pi_m(\alpha),\pi_m(\beta)) < 2L_\delta. \tag{4.7}$$

It follows that $d_P(\pi_\rho(P_0),\pi_\rho(P_1)) < c_1' 2L_\delta$, where c_1' is the constant from (4.3).

Let us prove (3).

Begin with $P_0 \in P(S)$ such that $d_P(P_0,\pi_\rho(P_0)) = R$, and consider a second pants decomposition P_1. Let $\rho' = \rho(P_0,P_1)$ be a hierarchy path from P_0 to P_1 and let m' be the main geodesic of ρ'. Let $P_i' = \pi_\rho(P_i)$. Let $v_i \in \pi_S(P_i')$ be vertices of P_i' that lie on m (see Figure 1).

In view of (4.3), it suffices to prove that a bound of the form $d_P(P_0,P_1) < R/C$ implies a uniform bound on $d_S(v_0,v_1)$ for R at least some R_0. In particular, if there is a uniform bound $d_S(v_0,v_1) < B$ then we have

$$d_S(P_0',P_1') < B+2$$

and therefore by (4.3)

$$d_P(\pi_\rho(P_0),\pi_\rho(P_1)) = d_P(P_0',P_1') < c_1'(B+2) \tag{4.8}$$

which is the desired conclusion for (3). We proceed to deduce this implication.

As in Lemma 4.2, we take $M_3 = M_1 + K + 2M_2 + 2$ and let c_1, c_2, be constants supplied by the distance formula of Theorem 2.6 such that

$$d_P(Q,Q') \asymp_{c_1,c_2} \sum_W' \left[[d_W(Q,Q')]\right]_{M_3} \tag{4.9}$$

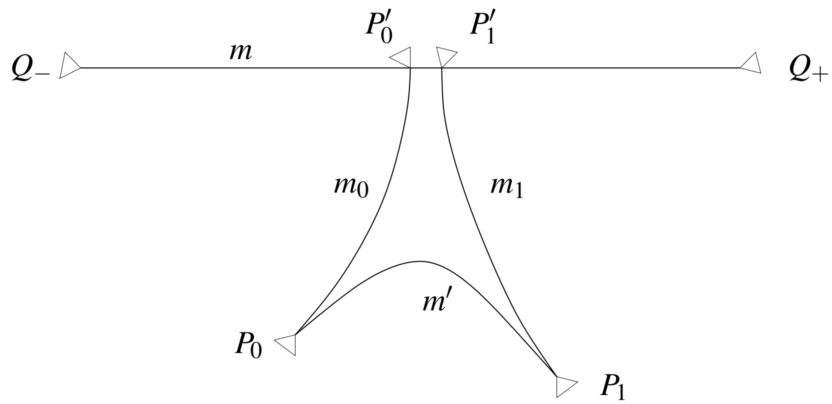

Figure 1: The main geodesics of the hierarchies in Theorem 4.3.

Let c_3 and c_4 be the constants determined by the distance formula for the threshold constant $2M_3$. Let m_i be the main geodesic of the hierarchy path $\rho_i = \rho(P_i, P'_i)$.

By the distance formula, once $d_P(P_0, P'_0) = R > 2c_3c_4$, we have

$$\sum\nolimits'_V [[d_V(P_0, P'_0)]]_{2M_3} \geq \frac{R}{c_3} - c_4 \geq \frac{R}{2c_3}.$$

It follows that either

$$d_S(P_0, P'_0) = |m_0| > R/4c_3 \tag{4.10}$$

or

$$\sum\nolimits'_{V \subsetneq S} [[d_V(P_0, P'_0)]]_{2M_3} > R/4c_3. \tag{4.11}$$

(The first corresponds to the $W = S$ term taking up at least half of the sum in the distance formula (4.9), while the second corresponds to the rest of the terms taking up at least half of the sum). By hyperbolicity of $\mathscr{C}(S)$, we have an A_S, B_S and C_S depending only on the hyperbolicity constant δ for $\mathscr{C}(S)$ so that provided $|m_0| > A_S$ we have

$$\operatorname{diam} \pi_m(\mathscr{N}^{\mathscr{C}}_{|m_0|/C_S}(P_0)) \leq B_S$$

where $\mathscr{N}^{\mathscr{C}}_r$ denotes an r-neighborhood with respect to the $\mathscr{C}(S)$-metric.

Choose the constants R_0, C, and N so that we have

$$R_0 > 4c_3A_S, \;\; C > 4c_3C_S, \;\text{ and }\; N > c'_1B_S.$$

Suppose first that (4.10) holds. Since

$$\pi_S(\mathscr{N}_r(P_0)) \subset \mathscr{N}^{\mathscr{C}}_r(P_0),$$

we may conclude that if $R > R_0$ and

$$d_P(P_0, P_1) \le \frac{R}{C} < \frac{|m_0|}{C_S}$$

then

$$d_S(v_0, v_1) \le B_S.$$

This conclues the proof in this case.

Now suppose that (4.10) does not hold, and thus (4.11) holds.

We claim that for some B just depending on S, if $d_S(v_0, v_1) \ge B$ then for any proper subsurface V of S we have

$$d_V(P_0, P_1) \ge d_V(P_0, P_0') - M_3. \tag{4.12}$$

To see this, we observe that if V is not a component domain of ρ_0 then the claim clearly holds since $M_3 > M_1$ and $d_V(P_0, P_0') < M_1$. Assume, then, that V is a component domain of ρ_0, so that ∂V has distance at most 1 from m_0.

As in (4.7), δ-hyperbolicity of $\mathscr{C}(S)$ guarantees that if there is a point v_0' of m_0 which is within distance 2 of a point v_1' of m_1 then we have $d_S(v_0, v_1) \le 2L_\delta$, since $v_0 = \pi_m(v_0')$ and $v_1 = \pi_m(v_1')$. Thus, if we choose $B > 2L_\delta$, then V cannot be a component domain of $\rho_1 = \rho(P_1, P_1')$ because then ∂V would be distance at most 1 from both m_0 and m_1. We conclude that $d_V(P_1, P_1') \le M_1$. Since

$$d_V(P_0', P_1') \le K + 2M_2$$

by the consequence (2.2) of K-bounded combinatorics, the triangle inequality gives the claim (4.12) by the choice of M_3.

Summing, we have

$$\begin{array}{rcl} \sum'_{V \subsetneq S} [[d_V(P_0, P_1)]]_{M_3} & \ge & \sum'_{V \subsetneq S} \left([[d_V(P_0, P_0')]]_{2M_3} - M_3 \right) \\ & \ge & \frac{1}{2} \sum'_{V \subsetneq S} [[d_V(P_0, P_0')]]_{2M_3} \end{array}$$

But then (4.11) gives the bound

$$c_1 d_P(P_0, P_1) + c_2 \ge \frac{R}{8c_3}. \tag{4.13}$$

Set $c_5 = 16c_3c_1$. Then for $R > 16c_3c_2$, we have

$$d(P_0, P_1) \ge R/c_5.$$

Choose the constants C, R_0 and N so that

$$C > c_5 + 4c_3 C_S,$$

$$R_0 > 16c_3c_2 + 2c_3c_4 + 4c_3A_S$$

and

$$N > \max\left\{c_1'(B+2), c_1'(2L_\delta + B_S)\right\}.$$

We have shown that for $R > R_0$, if

$$d_P(P_0, P_1) \leq R/C,$$

then

$$d_S(v_0, v_1) \leq B.$$

This completes the proof.

□

Theorem 4.3 implies that a hierarchy path joining points with uniformly bounded projection distances to all proper, non-annular subsurfaces of S produces a quasi-geodesic in $P(S)$, moreover a *stable* one.

Theorem 4.4. *For each K, K_0, there is a D so that if $\rho = \rho(Q_+, Q_-)$ is a hierarchy path such that for some K its endpoints satisfy the non-annular K-bounded combinatorics condition and if*

$$F : [0,T] \to P(S)$$

is a K_0-quasi-geodesic with $F(0) = Q_-$ and $F(T) = Q_+$, then we have

$$d_P(F(t), \pi_\rho(F(t))) \leq D. \tag{4.14}$$

That is, any quasi-geodesic in $P(S)$ with the same endpoints as ρ must lie within a bounded neighborhood of $|\rho|$, where the bound depends on the quality of the quasi-geodesic. This is proven using the usual Morse projection argument as in Mostow's rigidity theorem; see [MM1, Lemma 6.2].

4.2 Coboundedness

With Theorem 4.4 in hand, we return to the proof of Theorem 4.1 in the finite case. Namely, we show that a geodesic segment $\mathbf{g}$ both of whose endpoints project to $\mathscr{K}_0$, with Bers markings $\nu_\pm(\mathbf{g})$ associated to its endpoints that have bounded combinatorics, must be $\mathscr{K}$-cobounded for a suitable $\mathscr{K}$.

Using just bounded combinatorics on non-annular subsurfaces, we apply Theorem 4.4 to show that the pants decompositions that arise along $\mathbf{g}$ uniformly fellow travel, with respect to the Weil-Petersson metric, a hierarchy path joining Bers pants decompositions for the endpoints of the segment.

To conclude that $\mathbf{g}$ projects to a compact $\mathscr{K}$, we will require the bound on the annular projection distances $d_\gamma(\nu^+,\nu^-)$ as well.

Indeed, suppose there is a sequence of examples $\mathbf{g}_n$ with endpoints in $\mathscr{K}_0$ and uniformly bounded combinatorics (condition (4.1)), and a compact exhaustion $\{\mathscr{K}_n \subset \mathscr{M}(S)\}$ for which $\mathbf{g}_n$ exits $\widetilde{\mathscr{K}_n}$ (from now on we let $\widetilde{\mathscr{K}}$ and $\widetilde{\mathscr{K}_n}$ denote the preimages in $\operatorname{Teich}(S)$). Let $\mathbf{g}_n$ have endpoints X_n^+ and X_n^-. Let $\nu_n^+ = \nu(X_n^+)$ and $\nu_n^- = \nu(X_n^-)$ be the corresponding Bers markings at the endpoints and let $Q_n^\pm = \operatorname{base}(\nu_n^\pm)$ the corresponding Bers pants decompositions. Let $\rho_n = \rho(Q_n^+, Q_n^-)$ denote hierarchy paths associated to $Q_n^\pm$. By Theorem 2.1 $Q \circ \mathbf{g}_n$ is a quasigeodesic of uniform quality, so we obtain from (4.14) in Theorem 4.4 a constant D such that

$$d_P(Q(\mathbf{g}_n(t)), \pi_{\rho_n}(Q(\mathbf{g}_n(t)))) \leq D. \tag{4.15}$$

Fix ε_0 smaller than $\inf_{Z\in\mathscr{K}_0}(\operatorname{sys}(Z))$, and consider the length L_n of the longest interval J_n in the domain of $\mathbf{g}_n$ for which there is a curve $\gamma_n \in \mathscr{C}(S)$ with $\ell_{\mathbf{g}_n,\gamma_n}(t) \leq \varepsilon_0$ for $t \in J_n$. After passing to a subsequence, there are two cases:

Case 1: *The lengths L_n are unbounded.* Then there is a family of intervals $J_n = [a_n, b_n]$ and curves γ_n for which every point in $\mathbf{g}_n(J_n)$ is a bounded distance from the stratum $\mathscr{S}_{\gamma_n}$. Let x_n, $y_n \in \mathscr{S}_{\gamma_n}$ be the closest points in the stratum to the endpoints $\mathbf{g}_n(a_n)$ and $\mathbf{g}_n(b_n)$. Since strata are geodesically embedded in $\operatorname{Teich}(S)$ [Wol2] we have $d_{\mathscr{S}_{\gamma_n}}(x_n, y_n) \to \infty$. Applying Theorem 2.1 to $\mathscr{S}_{\gamma_n}$, which is naturally the Teichmüller space of the subsurface $W_n = S \setminus \gamma_n$, we find that

$$d_{P(W_n)}(Q(x_n), Q(y_n)) \to \infty.$$

The distance formula (5) of Theorem 2.6 implies that there exist (non-annular) subsurfaces $X_n \subseteq W_n$ such that $d_{X_n}(Q(x_n), Q(y_n)) \to \infty$.

Now since $d_{\mathrm{WP}}(x_n, \mathbf{g}_n(a_n))$ and $d_{\mathrm{WP}}(y_n, \mathbf{g}_n(b_n))$ are bounded, and since $Q(\mathbf{g}_n)$ is a bounded distance in $P(S)$ from $|\rho_n|$ (by (4.15), we find that there are i_n, j_n

such that $d_{X_n}(\rho_n(i_n),\rho_n(j_n)) \to \infty$. But by Theorem 2.6, part (4), this means that $d_{X_n}(Q_n^-,Q_n^+)$ is unbounded in n, contradicting the hypothesis (4.1).

Case 2: *The lengths L_n are bounded by some $L' > 0$.* In this case, we will argue that if the systole goes to 0 on $\mathbf{g}_n$ then Dehn twisting is building up somewhere along the geodesic and that this buildup persists to its endpoints. This conclusion will contradict the bounds on annulus projections:

Lemma 4.5. *Given positive constants ε_0, L and a, let $\mathbf{g}_n\colon [0,T_n] \to \mathrm{Teich}(S)$ be a sequence of Weil-Petersson geodesics of length $2a < T_n \le L$, and let $J_n \subset [a, T_n - a]$ be subintervals with the property that for each $\alpha \in \mathscr{C}(S)$ we have*

$$\max_{t\in J_n} \ell_{\mathbf{g}_n,\alpha}(t) \ge \varepsilon_0. \tag{4.16}$$

Then either

1. $\inf_n \inf_{t\in J_n} \mathrm{sys}(\mathbf{g}_n(t)) > 0$*, or*
2. *after possibly passing to a subsequence, there are $\gamma_n \in \mathscr{C}(S)$ for which*

$$\inf_{J_n} \ell_{\mathbf{g}_n,\gamma_n} \to 0$$

and

$$d_{\gamma_n}(\nu(\mathbf{g}_n(0)), \nu(\mathbf{g}_n(T_n))) \to \infty.$$

We postpone the proof of this lemma to §4.3, and use it now to complete the proof of Theorem 4.1 in the finite case.

Let $\mathbf{g}_n(s_n)$ be a sequence of points on $\mathbf{g}_n$ for which $\mathrm{sys}(\mathbf{g}_n(s_n)) \to 0$, and let $J_n = [a_n, b_n]$ be minimal-length intervals containing s_n such that, for each $\gamma \in \mathscr{C}(S)$, $\max_{J_n} \ell_{\mathbf{g}_n,\gamma} \ge \varepsilon_0$. Such intervals exist since $\mathrm{sys} > \varepsilon_0$ at the endpoints of $\mathbf{g}_n$, and since we are in Case 2, the length of a minimal one is at most L'.

Let $K_{\mathrm{W}P}$ and $C_{\mathrm{W}P}$ denote the multiplicative and additive quasi-isometry constants of Theorem 2.1, and choose $K' > K_{\mathrm{W}P}((D+4)c_1' + D + C_{\mathrm{W}P})$.

Let $I_n = [t_n^-, t_n^+]$ be the interval containing J_n satisfying either $t_n^+ - b_n = K'$ or t_n^+ equals the forward endpoint of $\mathbf{g}_n$ if the latter is distance less than K' from b_n, and similarly for t_n^- and a_n. In particular the length of I_n is bounded by $2K' + L'$. Note also that the distance of J_n from each endpoint of I_n is uniformly bounded below: in the case where t_n^+ or t_n^- is an endpoint of $\mathbf{g}_n$, this follows from the fact that those endpoints project to $\mathscr{K}_0$, and that ε_0 was chosen strictly smaller than $\inf_{Z\in\mathscr{K}_0}(\mathrm{sys}(Z))$.

We may therefore apply Lemma 4.5, where I_n play the role of the parameter intervals $[0, T_n]$, to conclude (possibly passing to a subsequence) the existence of curves γ_n for which

$$d_{\gamma_n}(\nu(\mathbf{g}_n(t_n^-)), \nu(\mathbf{g}_n(t_n^+))) \to \infty \tag{4.17}$$

and $t_n \in J_n$ such that

$$\ell_{\mathbf{g}_n,\gamma_n}(t_n) \to 0.$$

If $\mathbf{g}_n(t_n^+)$ is not an endpoint of $\mathbf{g}_n$, we have $d_{\mathrm{WP}}(\mathbf{g}_n(t_n), \mathbf{g}_n(t)) \geq K'$ for $t \geq t_n^+$, so we conclude

$$d_P(Q(\mathbf{g}_n(t_n)), Q(\mathbf{g}_n(t))) \geq K'/K_{\mathrm{WP}} - C_{\mathrm{WP}}.$$

Now by (4.15), we obtain the bound

$$d_P(Q(\mathbf{g}_n(t)), \pi_{\rho_n}(Q(\mathbf{g}_n(t)))) \leq D.$$

Hence, we have

$$d_P(Q(\mathbf{g}_n(t_n)), \pi_{\rho_n}(Q(\mathbf{g}_n(t)))) \geq K'/K_{\mathrm{WP}} - C_{\mathrm{WP}} - D. \tag{4.18}$$

Let $v_n(t)$ denote a vertex of the main geodesic of ρ_n which lies in $\pi_{\rho_n}(Q(\mathbf{g}_n(t)))$ (such a vertex exists by definition of π_{ρ_n}). Now since γ_n lies in $Q(\mathbf{g}_n(t_n)) = \mathrm{base}(\mu(\mathbf{g}_n(t_n))$ for large n (recalling that $\ell_{\mathbf{g}_n,\gamma_n}(t_n) \to 0$), by (4.18) together with Lemma 4.2 we get a lower bound on $\mathscr{C}(S)$-distance,

$$d_S(\gamma_n, v_n(t)) \geq \frac{1}{c_1'}\left(\frac{K'}{K_{\mathrm{WP}}} - C_{\mathrm{WP}} - D\right) \geq D + 4. \tag{4.19}$$

Now for $t \geq t_n^+$, again by (4.15), we can connect any vertex of $\mu(\mathbf{g}_n(t))$ to $v_n(t)$ by a path in $\mathscr{C}(S)$ of length at most $D+2$. Hence by (4.19) every vertex in this path has distance at least 2 from γ_n, and therefore intersects γ_n.

By the Lipschitz property of projections to $\mathscr{A}_{\gamma_n}$, Proposition 2.5, it follows that we have

$$d_{\gamma_n}(v_n(t), \mu(\mathbf{g}_n(t))) \leq 4(D+2) \tag{4.20}$$

for each $t \geq t_n^+$. Now, the diameter of the projection to $\mathscr{A}_{\gamma_n}$ of all the vertices of m_n that are forward of $v_n(t_n^+)$ is bounded above by $2M_2$, by Theorem 2.6, part 4. By the triangle inequality (applying (4.20) once for $t = t_n^+$ and once for $t > t_n^+$), we have

$$d_{\gamma_n}(\nu(\mathbf{g}_n(t)), \nu(\mathbf{g}_n(t_n^+))) < 8(D+2) + 2M_2 \tag{4.21}$$

for all $t \geq t_n^+$. The same bound can be obtained for t_n^- and $t \leq t_n^-$. Of course if t_n^+ is the forward endpoint of $\mathbf{g}_n$ then (4.21) holds trivially, and similarly for t_n^-.

Now applying these bounds to the endpoints $u_n \leq t_n^-$ and $w_n \geq t_n^+$ of $\mathbf{g}_n$, and using the growth inequality (4.17), we obtain

$$d_{\gamma_n}(\nu(\mathbf{g}_n(u_n)), \nu(\mathbf{g}_n(w_n))) \to \infty.$$

But this contradicts the bounded-combinatorics hypothesis on $\mathbf{g}_n$.

We conclude that in fact $\mathbf{g}_n$ are $\mathscr{K}$-cobounded for some $\mathscr{K}$. This concludes the proof of Theorem 4.1 in the case of finite intervals, modulo Lemma 4.5.

4.3 Proof of Lemma 4.5

We will apply Wolpert's discussion of limits of finite length geodesics in the Weil-Petersson completion ([Wol3, Theorem 23]):

Theorem 4.6. – Wolpert. (GEODESIC LIMITS) *Let $\mathbf{g}_n\colon [0,L] \to \overline{\mathrm{Teich}(S)}$ be a sequence of finite length geodesic segments of length L in the Weil-Petersson completion. Then there exists a partition of the interval $[0,L]$ by $0 = t_0 < t_1 < t_2 < \dots t_k < t_{k+1} = L$, and simplices $\sigma_0, \dots, \sigma_{k+1}$ and simplices $\tau_i = \sigma_{i-1} \cap \sigma_i$ in $\widehat{\mathscr{C}(S)}$ and a piecewise geodesic*

$$\hat{\mathbf{g}}\colon [0,L] \to \overline{\mathrm{Teich}(S)}$$

with the following properties.

1. *$\hat{\mathbf{g}}((t_{i-1}, t_i)) \subset \mathscr{S}_{\tau_i}$, $i = 1, \dots, k+1$,*
2. *$\hat{\mathbf{g}}(t_i) \in \mathscr{S}_{\sigma_i}$, $i = 0, \dots, k+1$,*
3. *there are elements $\psi_n \in \mathrm{Mod}(S)$ and $\mathscr{T}_{i,n} \in \mathrm{tw}(\sigma_i - \tau_i \cup \tau_{i+1})$, for $i = 1, \dots, k$, so that after passing to a subsequence, $\psi_n(\mathbf{g}_n([0,t_1]))$ converges in $\overline{\mathrm{Teich}(S)}$ to the restriction $\hat{\mathbf{g}}([0,t_1])$ and for each $i = 1, \dots, k$, and $t \in [t_i, t_{i+1}]$,*

   $$\mathscr{T}_{i,n} \circ \dots \circ \mathscr{T}_{1,n} \circ \psi_n(\mathbf{g}_n(t)) \to \hat{\mathbf{g}}(t)$$

   as $n \to \infty$.
4. *The elements ψ_n are either trivial or unbounded, and the elements $\mathscr{T}_{i,n}$ are unbounded.*

The piecewise-geodesic $\hat{\mathbf{g}}$ is the minimal length path in $\overline{\mathrm{Teich}(S)}$ joining $\hat{\mathbf{g}}(0)$ to $\hat{\mathbf{g}}(L)$ and intersecting the closures of the strata $\mathscr{S}_{\sigma_1}, \mathscr{S}_{\sigma_2}, \dots, \mathscr{S}_{\sigma_k}$ in order.

For convenience we define, for $i \geq 0$,

$$\varphi_{i,n} = \mathscr{T}_{i,n} \circ \ldots \circ \mathscr{T}_{1,n} \circ \psi_n. \tag{4.22}$$

To understand the meaning of this somewhat technical statement, it is helpful to focus on the case where all the τ_i are empty. In this case the statement is that the sequence of bounded-length geodesics is converging to a chain of segments in the interior of $\mathrm{Teich}(S)$ with endpoints on various strata. Moreover, in the approximating pictures, the geodesics approach the strata and "wind around" them in the sense that the twisting parameters for at least one curve per stratum grow without bound. This is encoded by the twists $\mathscr{T}_{i,n}$.

See Figure 2 for a cartoon of this limiting process.

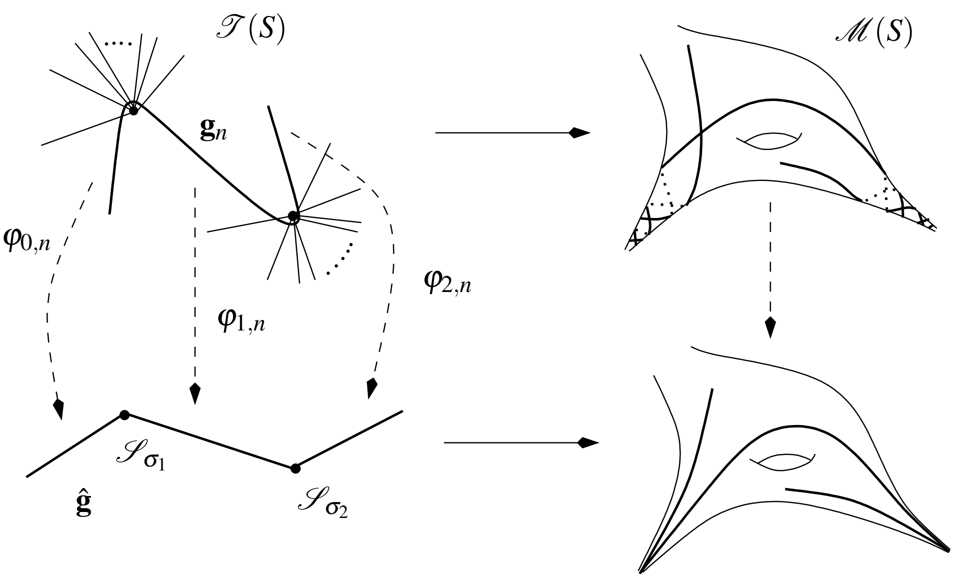

Figure 2: Geodesic limits in Teichmüller and Moduli space. Horizontal arrows denote the covering from Teichmüller to moduli space, and the vertical arrows denote convergence. In this figure, τ_i are all empty.

Now proceeding with the proof of Lemma 4.5, fix positive ε_0, L, and a. It suffices to show, for any sequence

$$\mathbf{g}_n \colon [0, T_n] \to \mathrm{Teich}(S)$$

of Weil-Petersson geodesics of length $T_n \leq L$, and intervals $J_n \subset [a, T_n - a]$ such that

1. $\sup_{t \in J_n} \ell_{\mathbf{g}_n,\alpha}(t) > \varepsilon_0$ for each $\alpha \in \mathscr{C}(S)$ and

2. $\inf_{t \in J_n} \mathrm{sys}(\mathbf{g}_n(t)) \to 0$,

that, after passing to a subsequence, there are $\gamma_n \in \mathscr{C}(S)$ such that

$$\inf_{t \in J_n} \ell_{\mathbf{g}_n,\gamma_n}(t) \to 0$$

and

$$d_{\gamma_n}(\nu(\mathbf{g}_n(0)), \nu(\mathbf{g}_n(T_n))) \to \infty.$$

Passing to a subsequence, trimming the intervals slightly and changing the constants, we may assume that $T_n \equiv L$, and that J_n converge to a subinterval J. Note that the lengths of J_n are bounded below since $\ell_{\mathbf{g}_n,\gamma_n}$ achieves the value ε_0 in J_n but its infimum goes to 0; hence J has positive length.

Then by Theorem 4.6, after passing again to a subsequence, we have a partition of the interval $[0,L]$ with $0 = t_0 < \ldots < t_k < t_{k+1} = L$, simplices $\sigma_0, \ldots, \sigma_{k+1}$, and $\tau_1, \ldots, \tau_{k+1}$ in the curve complex $\mathscr{C}(S)$, with $\tau_i \cup \tau_{i+1} \subset \sigma_i$ and a piecewise geodesic path

$$\hat{\mathbf{g}} \colon [0,L] \to \overline{\mathrm{Teich}(S)},$$

for which $\hat{\mathbf{g}}([t_j, t_{j+1}])$ is a geodesic segment in the stratum $\mathscr{S}_{\tau_{j+1}}$ joining the strata $\mathscr{S}_{\sigma_j}$ and $\mathscr{S}_{\sigma_{j+1}}$, and the elements $\mathscr{T}_{i,n} \in \mathrm{tw}(\sigma_i - \tau_i \cup \tau_{i+1})$ are unbounded in $\mathrm{Mod}(S)$. Assume the conclusions of Theorem 4.6 hold, and let $\varphi_{i,n}$ be as in (4.22).

For each i and n, let

$$\sigma_{i,n} = \varphi_{i,n}^{-1}(\sigma_i) = \varphi_{i-1,n}^{-1}(\sigma_i)$$

be the pullback of σ_i to the $\mathbf{g}_n$ picture. Similarly let

$$\tau_{i,n} = \varphi_{i-1,n}^{-1}(\tau_i).$$

We claim that $k > 0$ and in fact one of the t_i is contained in J. If not, then J is contained within some (t_{i-1}, t_i), and so $\hat{\mathbf{g}}(J) \subset \mathscr{S}_{\tau_i}$. Since $\inf_{J_n} \mathrm{sys}(\mathbf{g}_n) \to 0$, it follows that τ_i is nonempty. But this means that $\tau_{i,n}$ has length going to 0 at every point of $\mathbf{g}_n(J_n)$, which contradicts property (1).

Hence, we may fix positive $i \leq k$, such that $t_i \in J$, and let γ be a curve in $\sigma_i \setminus (\tau_i \cup \tau_{i+1})$ so that the power of the γ-Dehn twist $\mathscr{T}_\gamma$ determined by the element $\mathscr{T}_{i,n}$ is unbounded with n (since the multi-twist $\mathscr{T}_{i,n} \in \mathrm{tw}(\sigma_i - \tau_i \cup \tau_{i+1})$ is unbounded, there exists such a γ). Possibly passing to a subsequence, we can assume the power of $\mathscr{T}_\gamma$ tends to infinity. Let $\gamma_n \in \sigma_{i,n}$ denote the pullback

$$\gamma_n = \varphi_{i,n}^{-1}(\gamma).$$

For each $i = 0, \dots, k+1$, choose partial markings μ_i of S so that

1. $\sigma_i \subset \mathrm{base}(\mu_i)$, and
2. μ_i restricts to a full marking of each component $Y \subset S \setminus \sigma_i$ with complexity at least one.

Furthermore, for each $i = 0, \dots, k$, let μ_i^+ be an enlargement of μ_i so that $\mathrm{base}(\mu_i^+) = \mathrm{base}(\mu_i)$ and μ_i^+ restricts to a full marking of each component of $S \setminus \tau_{i+1}$ of complexity at least one. Likewise, for each $i = 1, \dots, k+1$, let μ_i^- be an enlargement of μ_i with $\mathrm{base}(\mu_i^-) = \mathrm{base}(\mu_i)$ and so that μ_i^- restricts to a full marking of each component of $S \setminus \tau_i$ of complexity at least one. Note that μ_i^+ differs from μ_i just by the addition of transversals to the components of $\sigma_i \setminus \tau_{i+1}$, and similarly for μ_i^- and $\sigma_i \setminus \tau_i$.

Further, define the pullbacks

$$\mu_{i,n}^+ = \varphi_{i,n}^{-1}(\mu_i^+)$$

and

$$\mu_{i,n}^- = \varphi_{i-1,n}^{-1}(\mu_i^-).$$

Now we want to measure the twisting of these markings relative to γ_n. We claim:

1. $d_{\gamma_n}(\mu_{i,n}^-, \mu_{i,n}^+) \to \infty$ as $n \to \infty$,
2. $d_{\gamma_n}(\mu_{j,n}^-, \mu_{j,n}^+)$ is bounded if $j \neq i$, and
3. $d_{\gamma_n}(\mu_{j,n}^+, \mu_{j+1,n}^-)$ is bounded for all j.

To see the first claim, note that

$$\varphi_{i,n}(\mu_{i,n}^-) = \mathscr{T}_{i,n}(\mu_i^-).$$

Thus, after applying $\varphi_{i,n}$ to all curves in our expression we get

$$d_\gamma(\mathscr{T}_{i,n}(\mu_i^-), \mu_i^+).$$

Now μ_i^- and μ_i^+ are fixed, and each contains γ as well as a transversal for γ. Since $\mathscr{T}_{i,n}$ contains an arbitrarily large power of $\mathscr{T}_\gamma$, claim (1) follows.

To see claim (2), note that $\mu_{j,n}^+$ and $\mu_{j,n}^-$ both contain

$$\mu_{j,n} = \varphi_{j,n}^{-1}(\mu_j) = \varphi_{j-1,n}^{-1}(\mu_j).$$

Observe further that $\mu_{j,n}$ contains $\sigma_{j,n}$, and in each component of $S \setminus \sigma_{j,n}$ it restricts to a full marking.

Now we claim that

$$\gamma_n \notin \sigma_{j,n} \text{ for any } j \neq i. \tag{4.23}$$

For otherwise the length of γ_n along $\mathbf{g}_n$ would converge to 0 both at t_i and at t_j, and hence by convexity on all of $[t_{i-1}, t_i]$ or $[t_i, t_{i+1}]$ (the first if $j < i$ and the second if $j > i$). This implies that $\gamma_n \in \tau_{i,n}$ or $\gamma_n \in \tau_{i+1,n}$, which contradicts the choice of $\gamma \in \sigma_i \setminus (\tau_i \cup \tau_{i+1})$, so we conclude that (4.23) holds. Thus, γ_n intersects $\mu_{j,n}$ nontrivially, so $\pi_{\mathscr{A}(\gamma_n)}(\mu_{j,n})$ is nonempty, and it follows that the projections of the two enlargements are a bounded distance apart in $\mathscr{A}(\gamma_n)$, establishing claim (2).

To prove claim (3), note that μ_j^+ and μ_{j+1}^- contain τ_{j+1} and restrict to full markings in $S \setminus \tau_{j+1}$, where their marking distance is some finite number. Hence we may connect them with a finite sequence of markings of the same type. Applying $\varphi_{j,n}^{-1}$, we obtain a sequence of the same length connecting $\mu_{j,n}^+$ to $\mu_{j+1,n}^-$, through markings that contain $\tau_{j+1,n}$ and are full in its complement. Since $\tau_{j+1,n}$ is contained in both $\sigma_{j,n}$ and $\sigma_{j+1,n}$, γ_n cannot lie in $\tau_{j+1,n}$ by (4.23). We conclude that all the markings intersect γ_n nontrivially, and this gives a bound on $d_{\gamma_n}(\mu_j^+, \mu_{j+1}^-)$, as desired.

Having established all three claims, we combine them with the triangle inequality to conclude

$$d_{\gamma_n}(\mu_{0,n}^+, \mu_{k+1,n}^-) \to \infty.$$

Now note that $\mu_{0,n}^+$ has bounded total length in $\mathbf{g}_n(0)$ and $\mu_{k+1,n}^-$ has bounded total length in $\mathbf{g}_n(L)$. It follows that

$$d_{\gamma_n}(\nu(\mathbf{g}_n(0)), \nu(\mathbf{g}_n(L))) \to \infty,$$

as desired.

It remains to check that

$$\inf_{J_n} \ell_{\mathbf{g}_n,\gamma_n} \to 0.$$

Recall that $t_i \in J = \lim J_n$, and $\ell_{\mathbf{g}_n,\gamma_n}(t_i) \to 0$. If $t_i \in J_n$ for n sufficiently large, then we are done, but even if not, note that on $[t_i, t_{i+1}]$ the length functions $\ell_{\mathbf{g}_n,\gamma_n}$

converge uniformly to $\ell_{\hat{\mathbf{g}},\gamma}$, and similarly for $[t_{i-1},t_i]$. Hence the infima on J_n converge to 0. This concludes the proof of Lemma 4.5.

4.4 The infinite case

We are left to consider the case when $\mathbf{g}$ is bi-infinite or the case of an infinite ray $\mathbf{r}$.

Suppose a ray $\mathbf{r}$ has its basepoint in $\widetilde{\mathscr{K}_0}$ and its ending lamination $\lambda = \lambda^+(\mathbf{r})$ has bounded combinatorics. In [MM2], it is shown that there exists an infinite hierarchy path $\rho_{\mathbf{r}}$ beginning at $Q(\mathbf{r}(0))$ so that $\rho_{\mathbf{r}}(i)$ is asymptotic to λ in $\pi(\mathscr{E}\mathscr{L}(S))$.

Letting $\mu_i = \rho_{\mathbf{r}}(i)$ be the markings along the hierarchy path $\rho_{\mathbf{r}}$, we may find points X_i in $\widetilde{\mathscr{K}_0}$ on which every curve in P_i has length bounded by some fixed ℓ, independent of i. Letting $X_0 = \mathbf{r}(0)$, the sequence of geodesic segments $\mathbf{g}_i = \mathbf{g}(X_0,X_i)$ joining X_0 to X_i projects to the compact set $\mathscr{K}$ by the above.

It follows that we may extract a limiting ray $\mathbf{r}_\infty$ in the visual sphere at X_0, which by Proposition 2.9 has ending lamination λ (as the lamination λ fills the surface). As each $\mathbf{g}_i$ lies in $\widetilde{\mathscr{K}}$, the limit $\mathbf{r}_\infty$ lies in $\widetilde{\mathscr{K}}$ as well. Then $\mathbf{r}_\infty$ is recurrent, and thus by the main theorem of [BMM] (Theorem 2.7 here) we have that $\mathbf{r}_\infty = \mathbf{r}$. We conclude that $\mathbf{r}$ lies in $\widetilde{\mathscr{K}}$ as desired.

Consider a bi-infinite geodesic $\mathbf{g}$, with ending laminations λ^+ and λ^- with K bounded combinatorics. In [MM2], it is shown that there exists $\rho_\pm = \rho(\lambda^+,\lambda^-)$, a bi-infinite hierarchy path limiting to $\lambda^+ \in \pi(\mathscr{E}\mathscr{L}(S))$ in the forward direction and $\lambda^- \in \pi(\mathscr{E}\mathscr{L}(S))$ in the backward direction. We choose X_i^+ and X_i^- in $\widetilde{\mathscr{K}_0}$ on which pants decompositions $P_i^+ \in |\rho_\pm|$ and $P_i^- \in |\rho_\pm|$ have bounded length, where $P_i^+ \to \lambda^+$ and $P_i^- \to \lambda^-$.

Again the geodesics $\mathbf{g}(X_0,X_i^+)$ and $\mathbf{g}(X_0,X_i^-)$ limit to rays $\mathbf{r}^+$ and $\mathbf{r}^-$ based at X_0, with ending laminations λ^+ and λ^- by Proposition 2.9 as above. Each of these rays lies in the set $\widetilde{\mathscr{K}}$ by the above, and is therefore recurrent. By the visibility property for recurrent rays, [BMM, Thm. 1.3], there is a unique bi-infinite ray $\mathbf{g}_\infty$ forward asymptotic to $\mathbf{r}^+$ and backward asymptotic to $\mathbf{r}^-$. It follows that $\mathbf{g}_\infty$ has the ending laminations λ^+ and λ^-, from which we conclude that $\mathbf{g}_\infty = \mathbf{g}$ by Theorem 2.7.

Now by an application of Theorem 4.4 to the quasi-geodesic $Q(\mathbf{g}(X_0,X_i^+))$, there is a $D > 0$ so that for each fixed $j > 0$, each pants decomposition P_j lies within distance D of $Q(\mathbf{g}(X_0,X_i^+))$ for each $i \geq j$. It follows that

$$d_{\mathrm{WP}}(X_j,\mathbf{g}(X_0,X_i^+)) < D' = K_{\mathrm{WP}}D + C_{\mathrm{WP}}$$

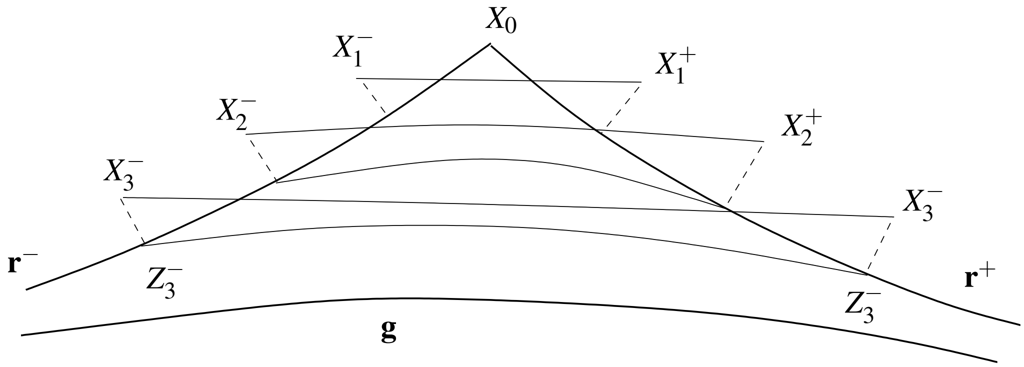

Figure 3: Extracting a bi-infinite geodesic limit.

for each $i \geq j$. Thus each X_j^+ lies distance at most D' from $\mathbf{r}^+$. Similarly each X_j^- lies distance at most D' from $\mathbf{r}^-$.

Thus, if Z_i^+ and Z_i^- are the nearest point projections of X_i^+ and X_i^- onto $\mathbf{r}^+$ and $\mathbf{r}^-$, the geodesic segments $\mathbf{g}(X_i^-, X_i^+)$ lie at a uniformly bounded distance from the geodesic $\mathbf{g}(Z_i^-, Z_i^+)$ since d_{WP} on Teich(S) is CAT(0). The geodesics $\mathbf{g}(Z_i^-, Z_i^+)$ converge to $\mathbf{g}$ by the visibility construction of [BMM, Thm. 1.3], so it follows that $\mathbf{g}(X_i^-, X_i^+)$ converges to $\mathbf{g}$ as well. But $\mathbf{g}(X_i^-, X_i^+)$ lies in $\widetilde{\mathscr{K}}$ for all $i > 0$, by the finite case of Theorem 4.1, so we may conclude that $\mathbf{g}$ also lies in $\widetilde{\mathscr{K}}$, completing the proof.

5 Counting closed orbits and topological entropy

We return to consider the Weil-Petersson geodesic flow on T^1Teich(S), the unit tangent bundle to Teich(S) and the flow on the quotient $\mathscr{M}^1(S)$. We recall from [BMM] that the geodesic flow is not everywhere defined, but is defined on the full Liouville-measure subset $\mathscr{F} \subset \mathscr{M}^1(S)$ consisting of lifts of bi-infinite geodesics to the unit tangent bundle.

Given any compact flow-invariant subset $\mathscr{K}$ of $\mathscr{M}^1(S)$, the question of the topological entropy $h_{\mathrm{top}}(\mathscr{K})$ of the flow φ^t can be formulated. In this section we show

Theorem 1.4. (TOPOLOGICAL ENTROPY) *There are compact flow-invariant subsets of $\mathscr{M}^1(S)$ of arbitrarily large topological entropy.*

The estimate of entropy follows directly from estimates on the asymptotic growth rate of the number of closed orbits of the geodesic flow in a compact set. Given a compact subset $\mathscr{K} \subset \mathscr{M}^1(S)$, let $n_{\mathscr{K}}(L)$ denote the number of closed orbits of the geodesic flow of length at most L that are contained in $\mathscr{K}$. We are interested in the asymptotic growth rate

$$p_\varphi(\mathscr{K}) = \liminf_{L\to\infty} \frac{\log n_{\mathscr{K}}(L)}{L}.$$

Theorem 1.5. (COUNTING ORBITS) *Given any $N > 0$, there is a compact Weil-Petersson geodesic flow-invariant subset $\mathscr{K} \subset \mathscr{M}^1(S)$ for which the asymptotic growth rate $p_\varphi(\mathscr{K})$ for the number of closed orbits in $\mathscr{K}$ satisfies*

$$p_\varphi(\mathscr{K}) \geq N.$$

The relationship between the conclusions of Theorem 1.5 and Theorem 1.4 for φ^t lies in Proposition 5.2, below, once we have shown that φ^t restricted to any compact invariant subset is *expansive*.

Definition 5.1. *A flow φ^t on a metric space (X,d) is* expansive *if there is a constant $\delta > 0$ so that the following property holds. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is any continuous surjective function with $f(0) = 0$ and such that $d(\varphi^t(x), \varphi^{f(t)}(x)) < \delta$ for all x,t. Then if x,y are such that*

$$d(\varphi^t(x), \varphi^{f(t)}(y)) < \delta$$

for all t, then there is a t_0 so that $\varphi^{t_0}(x) = y$.

We defer the proof that φ^t is expansive to Lemma 5.3 and proceed to the proof of Theorem 1.5, from which we will derive Theorem 1.4 as a consequence. To do so, we note the following.

Proposition 5.2. [KH] *Let φ^t be an expansive flow on a metric space (X,d), and let $\mathscr{K}$ be a compact invariant subset for φ. Then we have*

$$p_\varphi(\mathscr{K}) < h_{\mathrm{top}}(\mathscr{K}).$$

Proof of Theorem 1.5. We define a family of compact invariant subsets for the Weil-Petersson geodesic flow with a larger and larger exponential growth rate for the number of closed orbits of length at most L.

Given $K > 0$, we let $\mathscr{F}_K \subset T^1\mathrm{Teich}(S)$ denote the collection of lifts to $T^1\mathrm{Teich}(S)$ of bi-infinite geodesics in $\mathrm{Teich}(S)$ with K-bounded combinatorics; in other words if $\mathbf{g} \in \mathscr{F}_K$ then we have

$$d_Y(\lambda^+(\mathbf{g}), \lambda^-(\mathbf{g})) \leq K$$

for each proper essential subsurface $Y \subsetneq S$ that is not a three-holed sphere. Then by Theorem 4.1, $\mathscr{F}_K$ projects into a compact subset of $\mathscr{M}(S)$ and hence a compact subset of $\mathscr{M}^1(S)$, so the closure $\overline{\mathscr{F}}_K$ has compact image in its projection to $\mathscr{M}^1(S)$.

For K sufficiently large, the set $\mathscr{F}_K$ contains pseudo-Anosov axes by [DW], and thus its projection to $\mathscr{M}^1(S)$ contains closed orbits. Let $\mathscr{O}_K$ denote the collection of closed orbits in the projection of $\mathscr{F}_K$ to $\mathscr{M}^1(S)$. Then the closure $\overline{\mathscr{O}}_K$ is a compact geodesic-flow-invariant subset of $\mathscr{M}^1(S)$.

The asymptotic growth rate for the number of closed geodesics in $\mathscr{O}_K$ can estimated from below by a direct construction of a family of pseudo-Anosov elements of $\mathrm{Mod}(S)$ with K-bounded combinatorics.

We build this family using a construction of Thurston [Th, Thm. 7] as follows. A pair of (isotopy classes of) simple closed curves α and β *bind the surface* S if given representatives α^* and β^* on S for which $i(\alpha,\beta) = |\alpha^* \cap \beta^*|$ each component of $S \setminus (\alpha^* \cup \beta^*)$ is either a disk or an annulus that retracts to a boundary component of S.

The pair of curves determines a *Teichmüller disk*, $\Delta_{(\alpha,\beta)}$ an isometrically embedded copy of $\mathbb{H}^2$ (in the Teichmüller metric), and a representation ρ of the group $\langle \tau_\alpha, \tau_\beta \rangle$ generated by Dehn twists about α and β into the stabilizer of $\Delta_{(\alpha,\beta)}$ in $\mathrm{Mod}(S)$, which naturally acts isometrically on $\Delta_{(\alpha,\beta)}$. The representation ρ, which simply restricts the Dehn-twists as isometries of $\mathrm{Teich}(S)$ to the disk $\Delta_{(\alpha,\beta)}$, has the property that a given $\varphi \in \mathrm{Mod}(S)$ is of finite order, reducible, or pseudo-Anosov, according to whether it has image $\rho(\varphi)$ an elliptic, parabolic, or hyperbolic element of $\mathrm{PSL}_2(\mathbb{R})$.

In Thurston's construction, ρ sends the Dehn twists τ_α and τ_β to the elements

$$\rho(\tau_\alpha) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho(\tau_\beta) = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$$

where $k = i(\alpha,\beta)$ is the intersection number for the binding pair (α,β).

As the trace $\mathrm{tr}(\rho(\tau_\alpha \circ \tau_\beta^{-1}))$ is greater than 2, the element $\tau_\alpha \circ \tau_\beta^{-1}$ is pseudo-Anosov. Thurston observes, moreover, that for $k \geq 2$ the group $\rho(\langle \tau_\alpha, \tau_\beta \rangle)$ is free and ρ is faithful, so $\langle \tau_\alpha, \tau_\beta \rangle$ is free.

Given n positive integers $q_1, \ldots, q_n$, and letting

$$\psi_{(q_1,\ldots,q_n)} = \tau_\alpha^{q_1} \circ \tau_\beta^{-q_1} \circ \ldots \circ \tau_\alpha^{q_n} \circ \tau_\beta^{-q_n} \tag{5.1}$$

one may compute directly that $\mathrm{tr}(\rho(\psi_{(q_1,\ldots,q_n)}))$ is strictly greater than 2 and therefore that $\psi = \psi_{(q_1,\ldots,q_n)}$ is pseudo-Anosov.

Since $\langle \tau_\alpha, \tau_\beta \rangle$ is free, given $q_j \in [1,B]$ for $B > 1$, the conjugacy class of the element $\psi_{(q_1,\dots,q_n)}$ is uniquely determined by the n-tuple $\{q_1,\dots,q_n\}$ up to cyclic permutation, so the the number of distinct conjugacy classes of pseudo-Anosov mapping classes that arise from this construction is B^n/n.

We claim the combinatorics of the stable and unstable laminations for ψ are bounded in terms of B so there is a $K = K(B)$ for which the axes of all such pseudo-Anosov mapping classes lie in $\mathscr{F}_K$. To see this, note that the attracting and repelling fixed points for the hyperbolic element $\rho(\psi)$ in $\mathrm{PSL}_2(\mathbb{R})$ are real numbers with continued fraction expansion whose entries are bounded by kB (see [Ser]), and thus the axis projects into a compact subset of $\mathbb{H}^2/\rho(\langle \tau_\alpha, \tau_\beta \rangle)$ depending only on B. Since the inclusion $\Delta_{(\alpha,\beta)} \hookrightarrow \mathrm{Teich}(S)$ is an isometry for the Teichmüller metric, the axis for $\rho(\psi)$ includes as a geodesic into $\mathrm{Teich}(S)$ representing the invariant axis for ψ in the Teichmüller metric. As the axis projects to a compact subset of $\mathscr{M}(S)$, it follows from the main theorem of [Raf] that its stable and unstable laminations have bounded combinatorics, with bound $K = K(B)$ depending only on B.

By the upper bound on Weil-Petersson distance in Theorem 2.1, there is a constant C so that the Weil-Petersson translation distance of ψ is bounded above by nC. It follows that the asymptotic growth rate $p_\varphi(\overline{\mathscr{O}_K})$ of the number of closed geodesics in $\overline{\mathscr{O}_K}$ is bounded below by

$$\liminf_{n\to\infty} \frac{n\log(B) - \log(n)}{nC}$$

which tends to infinity with B. Thus the family of compact sets $\overline{\mathscr{O}_{K(B)}}$, has arbitrarily large asymptotic growth rates for their periodic orbits. $\square$

We now show the following.

Lemma 5.3. *The restriction of the Weil-Petersson geodesic flow to any compact invariant set is expansive.*

Proof. Let K be a compact invariant subset of $\mathscr{M}^1(S)$. There is $\delta = \delta(K) > 0$ such that if x and y lie in $\mathscr{M}^1(S)$ are a pair of points with $d(x,y) < \delta$ and $\tilde{x}$ is a lift of x to $T^1\mathrm{Teich}(S)$, then there exists a unique lift $\tilde{y}$ of y to $T^1\mathrm{Teich}(S)$, such that $d(\tilde{x},\tilde{y}) < \delta$.

Assume now we have a continuous surjective function $f\colon \mathbb{R} \to \mathbb{R}$ for which

$$d(\varphi^t(x), \varphi^{f(t)}(x)) < \delta.$$

Consider any x, y with the property that

$$d(\varphi^t(x), \varphi^{f(t)}(y)) < \delta$$

for all t.

Let $\tilde{\varphi}^t(x)$ be a lift of $\varphi^t(x)$, and for each t find the unique vector $w(f(t))$ in $T^1\mathrm{Teich}(S)$ such that

$$d(\tilde{\varphi}^t(x), w(f(t))) < \delta$$

and such that $\varphi^{f(t)}(y)$ is the projection of $w(f(t))$ to $\mathscr{M}^1(S)$. This gives a path $w(f(t))$ in $T^1\mathrm{Teich}(S)$. Since its projection to $\mathscr{M}^1(S)$ yields a geodesic in $\mathscr{M}(S)$, it follows that $w(f(t))$ projects to a geodesic in $\mathrm{Teich}(S)$.

As $w(f(t))$ and $\tilde{\varphi}^t(x)$ remain at uniformly bounded distance in $T^1\mathrm{Teich}(S)$, we conclude that their projections to $\mathrm{Teich}(S)$ remain at bounded distance as well.

Since distinct bi-infinite Weil-Petersson geodesics in $\mathrm{Teich}(S)$ diverge, in either forward or backward time, we conclude that $\tilde{\varphi}^t(x)$ and $w(f(t))$ are parametrizations by arclength of the same geodesic, and thus we may conclude that there is a t_0 for which

$$\varphi^{t_0}(x) = y.$$

It follows that the restriction of the flow to K is expansive.

□

Proof of Theorem 1.4. Theorem 1.4 follows immediately as a direct consequence of Theorem 1.5, Lemma 5.3 and Proposition 5.2. □

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